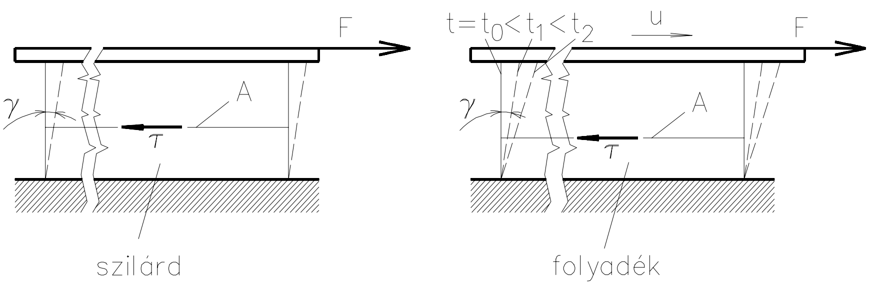
FLUID MECHANICS

1. Division of Fluid Mechanics

	Hydrostatics	Aerostatics	Hydrodynamics	Gasdynamics
v velocity				
p pressure				
ρ density				

2. Properties of fluids

Comparison of solid substances and fluids



solid fluid

$\tau = F/A$ [Pa] shear stress

Solid	γ (deformation) is proportional to τ shear stress
Fluids (Newtonian)	$d\gamma/dt$ (rate of deformation, strain rate) is proportional to τ shear stress

non-Newtonian fluids

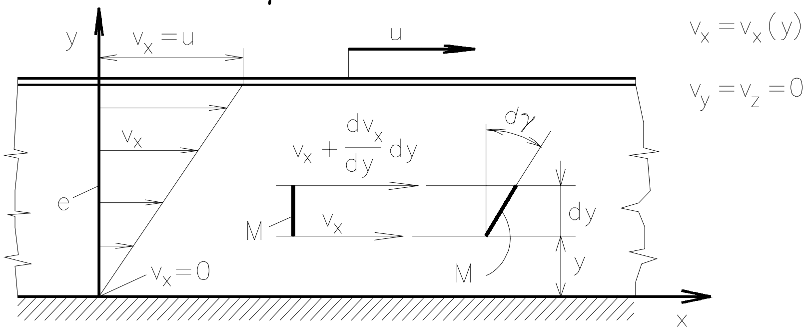
Fluids:

- no slip condition
- no change in internal structure at any deformation
- continuous deformation when shear stress exists
- no shear stress in fluids at rest

Viscosity

Velocity distribution: line or surface connecting the tips of velocity vectors the foot-end of which lies on a straight line or on a plane.

Turn of the bar M: dγ

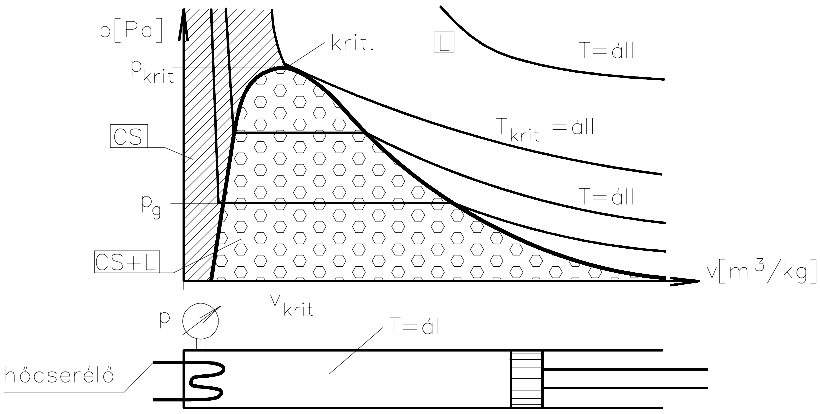


$$\frac{d\gamma}{dt} = \frac{dv_x}{dy}. \qquad \boxed{\tau_{xy} = \mu \frac{dv_x}{dy} = \mu \frac{d\gamma}{dt}.}$$ Newton's law of viscosity

$$[\mu] = [\tau]\left[\frac{dy}{dv_x}\right] = \frac{kgm}{s^2 m^2}\frac{m}{m/s} = \frac{kg}{ms}.$$ Dynamic viscosity

$$\boxed{\nu = \frac{\mu}{\rho}} \quad [m^2/s]$$ Kinematic viscosity

Compression of water vapor



heat exchanger T=const

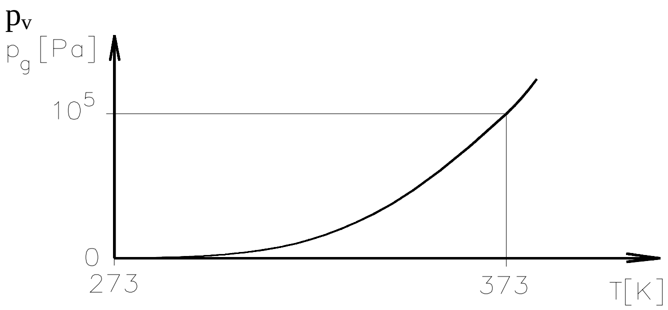
If $T >> T_{krit}$: gas O_2 and N_2 $\rightarrow T_{krit}$ 154 [K] and 126 [K]

$$pv = \boxed{\frac{p}{\rho} = RT}$$ ideal-gas law

where p[Pa], ρ [kg/m^3], T [K], $R = R_u / M$, R_u = 8314.3 J/kmol/K universal gas constant, M kg/kmol molar mass, for air: M=29 [kg/kmol], therefore R=287J/kg/K.

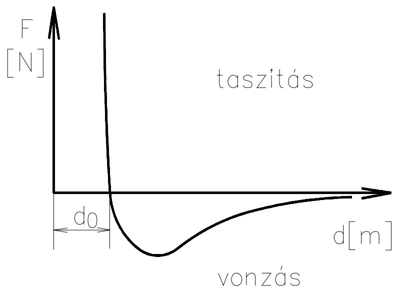
Cavitation

saturated steam pressure (vapor pressure) - temperature. Water 15 ^{0}C, p_v = 1700Pa, 100 ^{0}C, p_v = 1.013*10^5 Pa standard atmospheric pressure



Cavitation erosion

Interactions between molecules (attraction and repulse)



repulsion

attraction

Comparison f liquids and gases

	liquids	gases
distance between molecules	small ≅ d_0	large ≅ 10d_0
role of interactions of molecules	significant⇒ free surface	small ⇒ fill the available space
effect of change of pressure on the volume	small ⇒ 1000 bar causes 5% decrease in V	large⇒ in case of T=const V proportional to 1/p
cause of viscosity	attraction among molecules	momentum exchange among molecules
relation between viscosity and temperature pressure	T increases μ decreases independent	T increases μ increases independent

Comparison of real and perfect fluids

	real fluids	perfect fluids
viscosity	viscous	inviscid
density	compressible	incompressible
structure	molecular	continuous

3. Description of flow field

Scalar fields

Density $\rho_v = \lim_{\Delta V \Rightarrow \varepsilon^3} \frac{\Delta m}{\Delta V} \left[kg/m^3 \right]$ ΔV incremental volume $\varepsilon >> \lambda$ (mean free path)

continuum $\rho=\rho(\underline{r},t)$ $\rho=\rho(x,y,z,t)$

Pressure

$p = |\Delta \underline{F}| / |\Delta \underline{A}|$ [N/m^2], [Pa].

$p=p(\underline{r},t)$, $p=p(x,y,z,t)$

Temperature

$T=T(\underline{r},t)$

Vector fields

Velocity

$\underline{v} = \underline{v}(\underline{r}, t)$ Eulerian description of motion

Fields (of force) $[\underline{g}] = N/kg = m/s^2$.

gravity field: $\underline{g} = -g_g \underline{k}$ $g_g = 9.81$ N/kg

field of inertia: accelerating coordinate system ($\underline{a} = a\underline{i}$) $\underline{g_t} = -a\underline{i}$.

centrifugal field: rotating coordinate system $\underline{g_c} = \underline{r}\omega^2$

Characterization of fields

Characterization of scalar fields:

$$\mathrm{grad}p = \frac{\partial p}{\partial x}\underline{i} + \frac{\partial p}{\partial y}\underline{j} + \frac{\partial p}{\partial z}\underline{k} = \frac{\partial p}{\partial \underline{r}}$$ gradient vector

4 characteristics of the vector:

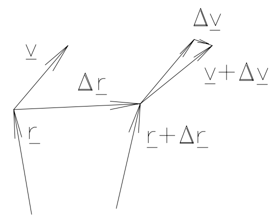
it is parallel with the most rapid change of p
it points towards increasing p
its length is proportional to the rate of the change of p
it is perpendicular to p = constant surfaces

Change of a variable: e.g. increment of pressure

$$\Delta p = p_B - p_A \cong \mathrm{grad}p\Delta\underline{s} = \frac{\partial p}{\partial x}\Delta x + \frac{\partial p}{\partial y}\Delta y + \frac{\partial p}{\partial z}\Delta z$$

Characterization of vector fields: $\underline{v} = v_x \underline{i} + v_y \underline{j} + v_z \underline{k} = \underline{v}(\underline{r}, t)$.

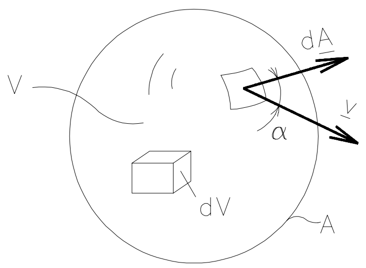
$v_x = v_x(x, y, z, t), v_y = v_y(x, y, z, t), v_z = v_z(x, y, z, t)$. vector field = 3 scalar fields



$$\Delta v_x \cong \text{grad} v_x \Delta \underline{r} = \frac{\partial v_x}{\partial x}\Delta x + \frac{\partial v_x}{\partial y}\Delta y + \frac{\partial v_x}{\partial z}\Delta z .$$

$$\Delta \underline{v} \cong \begin{bmatrix} \frac{\partial v_x}{\partial x}\Delta x + \frac{\partial v_x}{\partial y}\Delta y + \frac{\partial v_x}{\partial z}\Delta z \\ \frac{\partial v_y}{\partial x}\Delta x + \frac{\partial v_y}{\partial y}\Delta y + \frac{\partial v_y}{\partial z}\Delta z \\ \frac{\partial v_z}{\partial x}\Delta x + \frac{\partial v_z}{\partial y}\Delta y + \frac{\partial v_z}{\partial z}\Delta z \end{bmatrix}$$

Divergence: $\text{div}\underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$,



$$dq_v = \underline{v}d\underline{A} = |\underline{v}||d\underline{A}|\cos\alpha \,[m^3/s]$$

$$\boxed{\int_A \underline{v}d\underline{A} = \int_V \text{div}\underline{v}dV}$$ Gauss-Osztrogradszkij theorem

Rotation, vorticity: $$\text{rot}\underline{v} = \underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{bmatrix}$$

$$\boxed{\text{rot}\underline{v} = 2\underline{\Omega}}.$$

$$\boxed{\Gamma = \oint_G \underline{v}d\underline{s} = \int_A \text{rot}\underline{v}d\underline{A}}$$ Stokes theorem

Potential flow

$\underline{v} = \text{grad}\varphi$ condition: $\Gamma = \oint_G \underline{v} d\underline{s} = 0$, or $\text{rot}\underline{v} = 0$

Example: fields of force

for gravity force $\oint_G \underline{g} d\underline{s} = 0$ work of the field

U [m^2/s^2] potential of the field

$\boxed{\underline{v} = -\text{ grad U}}$

<u>gravity field</u>: $\underline{g} = -g_g \underline{k}$ $\boxed{U_g = g_g z + \text{konst.}}$

<u>field of inertia</u>: accelerating coordinate system ($\underline{a} = a\underline{i}$) $\underline{g_t} = -a\underline{i}$ $\boxed{U_t = ax + \text{konst.}}$

<u>centrifugal field</u>: rotating coordinate system $\underline{g_c} = \underline{r}\omega^2$ $\boxed{U_c = -\frac{r^2\omega^2}{2} + \text{kost.}}$

4. Kinematics

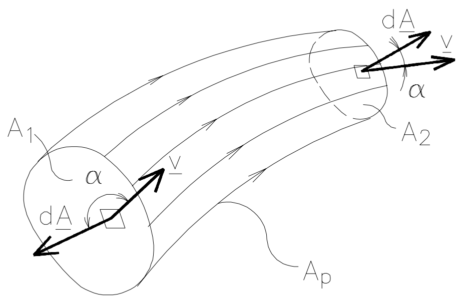
Definitions

<u>Pathline</u>: loci of points traversed by a particle (photo: time exposure)

<u>Streakline</u>: a line whose points are occupied by all particles passing through a specified point of the flow field (snapshot). Plume arising from a chimney, oil mist jet past vehicle model

<u>Streamline:</u> <u>v</u> x d<u>s</u> = 0 velocity vector of particles occupying a point of the streamline is tangent to the streamline.

<u>Stream surface</u>, stream tube: no flow across the surface.



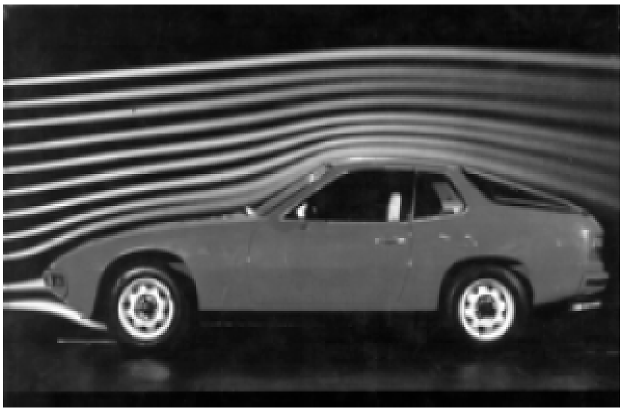
Time dependence of flow: Unsteady flow: **<u>v</u> = <u>v</u>(<u>r</u>, t)** Steady flow: **<u>v</u> = <u>v</u>(<u>r</u>)**

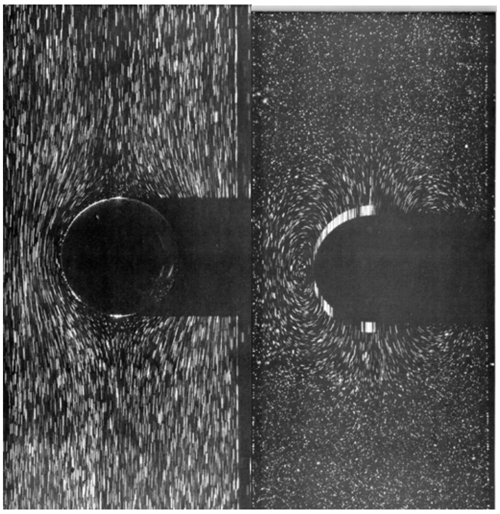
In some cases the time dependence can be eliminated through transformation of coordinate system.

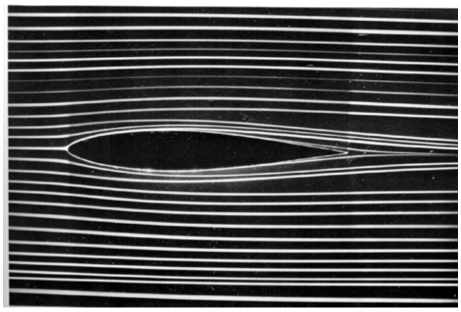
In steady flows pathlines, streaklines and streamlines coincide, at unsteady flows in general not.

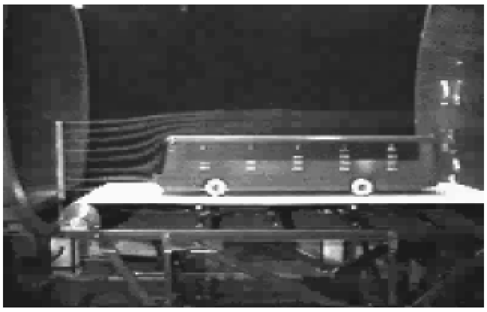
Flow visualization:
quantitative and/or qualitative information

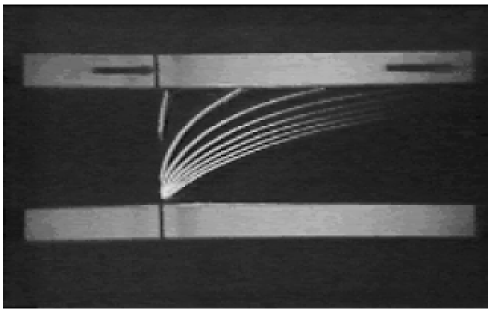
a) Transparent fluids, light-reflecting particles (tracers) moving with the fluid: particles of the same density, or small particles (high aerodynamic drag). Oil mist, smoke, hydrogen bubbles in air and in water, paints, plastic spheres in water, etc. PIV (Particle Image Velocymetry), LDA Laser Doppler Anemometry),
b) Wool tuft in air flow shows the direction of the flow.

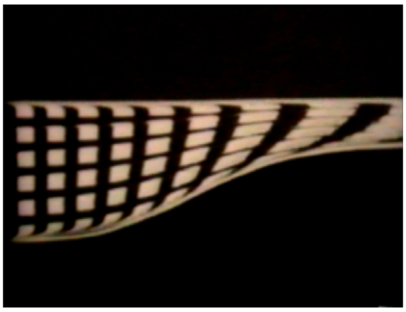


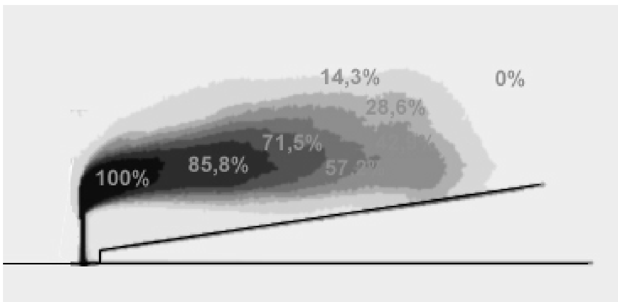


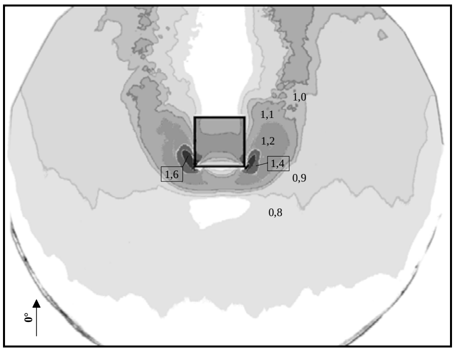




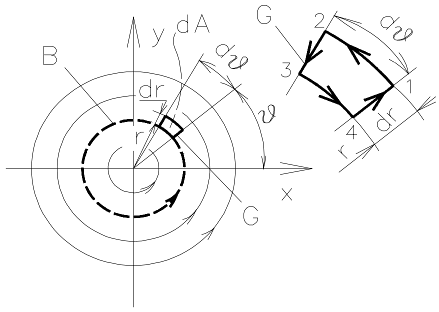








Irrotational (potential) vortex



Concept pf two-dimensional (2D), plane flows:

$$v_z = 0 \quad \text{and} \quad \frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = 0.$$

Because of continuity consideration at vortex flow v = v(r)   **v(r) = ?**

Calculation of rot$\underline{v}$ using Stokes theorem: $\boxed{\Gamma = \oint_G \underline{v} d\underline{s} = \int_A \text{rot}\underline{v} d\underline{A}}$

$$\oint_G \underline{v} d\underline{s} = \int_1^2 \underline{v} d\underline{s} + \underbrace{\int_2^3 \underline{v} d\underline{s}}_{=0} + \int_3^4 \underline{v} d\underline{s} + \underbrace{\int_4^1 \underline{v} d\underline{s}}_{=0}$$

Since v⊥ds at 2nd and 4th integrals, and at 1st and 3rd integral $\underline{v}$ and d$\underline{s}$ include an angle of 0^0 and 180^0:

$$\oint_G \underline{v} d\underline{s} = (r + dr) d\vartheta v(r + dr) - r d\vartheta v(r)$$

Since

$$v(r + dr) = v(r) + \frac{dv}{dr} dr$$

after substitution

$$\oint_G \underline{v} d\underline{s} = r d\vartheta \frac{dv}{dr} dr + dr d\vartheta v(r) + \underbrace{dr d\vartheta \frac{dv}{dr} dr}_{\approx 0}$$

In plane flow only (rot$\underline{v}$)$_z$ differs from 0.

$$\int_{dA} rot\underline{v} d\underline{A} = (rot\underline{v})_z r d\vartheta dr$$

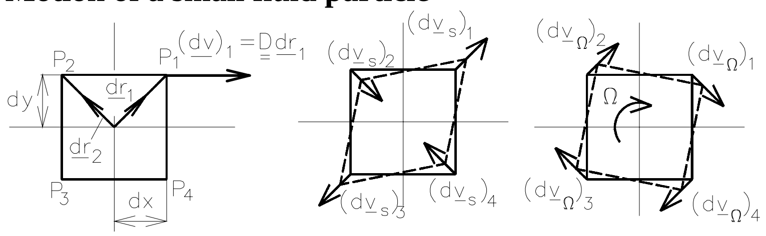
$$\boxed{(rot\underline{v})_z = \frac{dv}{dr} + \frac{v}{r}}.$$

Example: $v = \omega \cdot r \Rightarrow (rot\underline{v})_z = 2\omega$

In case of rot$\underline{v}$ = 0

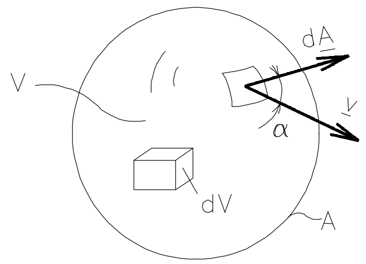
$\frac{dv}{v} = -\frac{dr}{r} \Rightarrow \ln v = -\ln r + \ln \text{Konst.} \Rightarrow \boxed{v = \frac{K}{r}}$. Velocity distribution in an irrotational (potential) vortex.

Motion of a small fluid particle



The motion of a FLID particle can be put together from **parallel shift, deformation and rotation.** In case of potential flow no rotation occurs.

5. Continuity equation



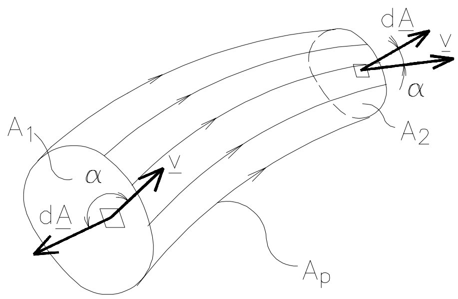
$dq_m = \rho \underline{v} d\underline{A} = \rho |\underline{v}||d\underline{A}| \cos\alpha \; [kg/s]$

integral form of continuity equation: $\boxed{\int_A \rho \underline{v} d\underline{A} + \int_V \frac{\partial \rho}{\partial t} dV = 0}$

differential form: $\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0}$, if the flow is steady: $\underline{v} = \underline{v}(\underline{r}) \Rightarrow$ $\boxed{\operatorname{div}(\rho \underline{v}) = 0}$,

if the fluid is incompressible ρ= const. $\boxed{\operatorname{div}\underline{v} = 0}$

Application of continuity equation for a stream tube
Steady flow, no flow across the surface.



Integral form of continuity equation for steady flow: $\int_A \rho \underline{v} d\underline{A} = 0$. "A" consists of the mantle A_p ($\underline{v} \perp d\underline{A}$) and A_1 and A_2 in- and outflow cross sections. $\int_{A_1} \rho \underline{v} d\underline{A} + \int_{A_2} \rho \underline{v} d\underline{A} = 0$. Since $\underline{v} d\underline{A} = |\underline{v}||d\underline{A}| \cos\alpha$, $\int_{A_1} \rho |\underline{v}||d\underline{A}| \cos\alpha + \int_{A_2} \rho |\underline{v}||d\underline{A}| \cos\alpha = 0$ Assumptions: over A_1 and A_2 ($v \perp A$) and over A_1 $\rho = \rho_1$ =const., over A_2 $\rho = \rho_2$ =const'. $\rho\bar{v}A$ = Const., where $\bar{v}$ mean velocity at changing cross section of a pipeline: $\boxed{\rho_1\bar{v}_1A_1 = \rho_2\bar{v}_2A_2}$ $\Rightarrow$ $\bar{v}_2 = \bar{v}_1 \dfrac{\rho_1 D_1^2}{\rho_2 D_2^2}$

6. Hydrostatics

Static fluid: forces acting on the mass (e.g. gravity) and forces acting over the surface (forces caused by pressure and ~~shear stresses~~) balance each other (no acceleration of fluid).

$$\rho\, dx\, dy\, dz\, g_x + dy\, dz\, p(x) - dy\, dz\left(p(x) + \frac{\partial p}{\partial x}dx\right) = 0$$

$\rho\, g_x = \dfrac{\partial p}{\partial x}$ $\Rightarrow$ $\boxed{\text{grad}\, p = \rho\, \underline{g}}$ fundamental equation of hydrostatics.

Assumption: $\underline{g} = -\text{grad}\, U$ (potential field of force)

$\text{grad}\, p = -\rho\, \text{grad}\, U$ $\Rightarrow$ p=const. surfaces coincide with U = Const. (equipotential surfaces)

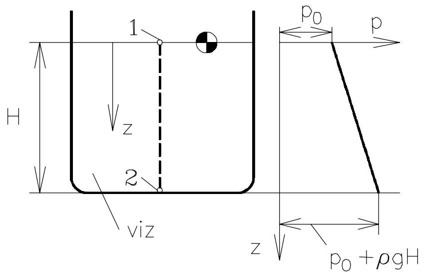
The surface of a liquid coincides with one of the U = Const. equipotential surfaces $\Rightarrow$ the surface is perpendicular to the field of force.

Assumptions $\underline{g} = -\text{grad}\, U$ (potential field of force), ρ = const. (incompressible fluid)

$$\frac{1}{\rho}\text{grad}\, p = \text{grad}\,\frac{p}{\rho} = -\,\text{grad}\, U \Rightarrow \text{grad}\left(\frac{p}{\rho} + U\right) = 0 \Rightarrow \boxed{\frac{p}{\rho} + U = \text{const.}}$$

$\boxed{\dfrac{p_1}{\rho_1} + U_1 = \dfrac{p_2}{\rho_2} + U_2}$ incomplete Bernoulli equation

Pressure distribution in a static and accelerating tank



$\underline{g}_g = g\underline{k}$, where $g = 9.81\,\text{N/kg}$. $\dfrac{\partial p}{\partial x}\underline{i} + \dfrac{\partial p}{\partial y}\underline{j} + \dfrac{\partial p}{\partial z}\underline{k} = \rho g \underline{k}$ $dp/dz = \rho g$, ρ=áll. $p = \rho g z + \text{Const.}$

If $z = 0$, then $p = p_0$. $\Rightarrow$ $\text{Const.} = p_0$ $\Rightarrow$ $\boxed{p = p_0 + \rho g z}$. In z = H point $p = p_0 + \rho g H$

$\boxed{\dfrac{p_1}{\rho_1} + U_1 = \dfrac{p_2}{\rho_2} + U_2}$ point 1 on the surface =0), point 2 at the bottom (z = H). At z coordinate pointing downwards $U = -gz$, $p_1 = p_0, z_1 = 0,$ $p_2 = ?, z_2 = H$.

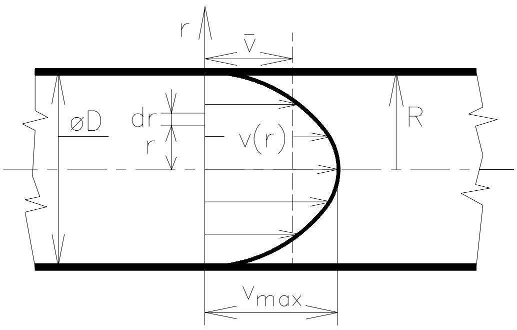
$\boxed{p_2 - p_0 = \rho g H}$

If the tank accelerates upwards, the fluid is static only in an upwards accelerating coordinate system. Here additional (inertial) field of force should be considered: $g_i = a\,\underline{k}$

$U_i = -az \quad U = U_g + U_i = -(g+a)z$. After substitution:

$$\boxed{p_2 - p_0 = \rho(g+a)H}$$

7. Calculation of mean velocity in a pipe of circular cross section



$\bar{v}$ = ? mean velocity

In cross section of diameter D the velocity distribution is described by a paraboloid. The difference of v_{max} and v(r) depends on the nth power of r $v(r) = v_{max}\left[1-(r/R)^n\right]$.

Mean velocity: $\bar{v} = \dfrac{4q_v}{D^2\pi}[m/s]$ where $q_v[m^3/s]$ is the flow rate.

The flow rate through an annulus of radius r thickness dr, cross section 2rπdr is dq_v = 2rπ v($\underline{r}$)dr $\Rightarrow$ $q_v = \int_0^R 2r\pi v_{max}\left[1-(r/R)^n\right]dr$.

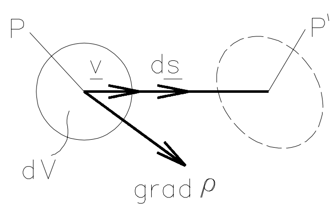
Integration yields: $q_v = R^2\pi v_{max}\dfrac{n}{n+2}$, so the mean velocity is:

$$\bar{v} = \frac{n}{n+2}v_{max}.$$

In case of paraboloid of 2^{nd} degree (n = 2) the mean velocity is half of the maximum velocity.

8. Local and convective change of variables

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\underline{v}) = 0 \Rightarrow \frac{\partial\rho}{\partial t} + \underline{v}\,\mathrm{grad}\,\rho + \rho\,\mathrm{div}(\rho\underline{v}) = 0$$



In point P the velocity is $\underline{v}$, the variation of density in space is characterized by gradρ. Unsteady flow: $\partial\rho/\partial t \neq 0$. Variation of density d$\rho$ in time dt?

Two reasons for variation of ρ:

a) Because of time dependence of density ($\partial\rho/\partial t \neq 0$), the variation of density in point P: $d\rho_l = \frac{\partial\rho}{\partial t}dt$

b) In dt time the fluid particle covers a distance $d\underline{s} = \underline{v}dt$ and gets in P' point, where the density differs $d\rho_c = \text{grad}\rho\ \underline{ds} = \text{grad}\rho\ \underline{v}\ dt$ from that of in point P.

$d\rho_l$ local variation of density (only in unsteady flows)

$d\rho_c$ convective variation of density is caused by the flow and the spatial variation of the density

The substantial variation of the density is time dt: $d\rho = d\rho_l + d\rho_c = \frac{\partial\rho}{\partial t}dt + \underline{v}\ \text{grad}\rho\ dt$,

The variation in time unit: $\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \underline{v}\text{grad}\rho \Rightarrow \frac{d\rho}{dt} + \rho\text{div}\underline{v} = 0$

9. Acceleration of fluid particles

The variation of v_x in unit time.

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \underline{v}\,\text{grad}v_x\,.$$

Acceleration of fluid particle in x direction.

The first term: local acceleration, the second term: convective acceleration.

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}$$
$$\frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}$$
$$\frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}$$

Determining the differential of $\underline{v}(\underline{r},t)$: $d\underline{v} = \frac{\partial\underline{v}}{\partial t}dt + \frac{\partial\underline{v}}{\partial\underline{r}}\frac{\partial\underline{r}}{\partial t}dt$. Referring $d\underline{v}$ to unit time, i.e. dividing it by dt: $\frac{d\underline{v}}{dt} = \frac{\partial\underline{v}}{\partial t} + \frac{\partial\underline{v}}{\partial\underline{r}}\frac{\partial\underline{r}}{\partial t}$, where $\frac{\partial\underline{r}}{\partial t} = \underline{v}$

Local acceleration is different from 0 if the flow is unsteady. The convective acceleration exists, if the magnitude and/or direction of flow alter in the direction of the motion of the fluid.

The formula for acceleration can be transformed:

$$\frac{d\underline{v}}{dt} = \frac{\partial\underline{v}}{\partial t} + \text{grad}\frac{v^2}{2} - \underline{v}\times\text{rot}\underline{v}\,.$$

10. Euler's equation (differential momentum equation)

Inviscid flow: $\boxed{\mu = 0}$

Resultant of forces = mass · acceleration
Inviscid flow: forces caused by the pressure and field of force.

In x direction:

$$\rho\,dx\,dy\,dz\,\frac{dv_x}{dx} = \rho\,dx\,dy\,dz\,g_x + p\,dy\,dz - (p + \frac{\partial p}{\partial x}dx)\,dy\,dz$$

$$\frac{dv_x}{dx} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\boxed{\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \mathrm{grad}\frac{\underline{v}^2}{2} - \underline{v}\times\mathrm{rot}\,\underline{v} = \underline{g} - \frac{1}{\rho}\mathrm{grad}\,\underline{p}}$$

If $\rho = \rho(p)$, $\quad -\frac{1}{\rho(p)}\mathrm{grad}\,p = -\mathrm{grad}\int_{p_0}^{p}\frac{dp}{\rho(p)}$

If ρ = const. the unknown variables are: v_x, v_y, v_x, p

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

11. Bernoulli equation

Inviscid flow: $\boxed{\mu = 0}$

$$\underbrace{\int_1^2 \frac{\partial \underline{v}}{\partial t}d\underline{s}}_{I} + \underbrace{\int_1^2 \mathrm{grad}\frac{v^2}{2}d\underline{s}}_{II} - \underbrace{\int_1^2 \underline{v}\times\mathrm{rot}\,\underline{v}\,d\underline{s}}_{III} = \underbrace{\int_1^2 \underline{g}\,d\underline{s}}_{IV} - \underbrace{\int_1^2 \frac{1}{\rho}\mathrm{grad}\,p\,d\underline{s}}_{V}$$

a) Since $\int_1^2 \mathrm{grad}\,f\,d\underline{s} = f_2 - f_1$ integral II $= \frac{v_2^2 - v_1^2}{2}$

b) If $\underline{g}$ = - gradU integral IV = – (U_2– U_1)

c) In case of steady flow ($\frac{\partial \underline{v}}{\partial t} = \underline{0}$) integral I =0

d) intergral III = 0, if

- $\underline{v}$ = 0 static fluid
- rot$\underline{v}$ = 0 potential flow
- d$\underline{s}$ lies in the plane determined by $\underline{v}$ and rot$\underline{v}$ vectors
- d$\underline{s}$ || $\underline{v}$ integration along streamlines
- d$\underline{s}$ || rot$\underline{v}$ integration along vortex lines

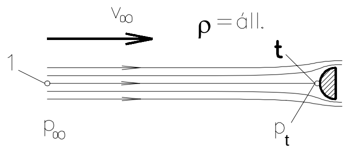
e) If $\rho = \text{const.}$ integral V = $-\frac{p_2 - p_1}{\rho}$, if $\rho = \rho(p)$, integral V = $\int_{p_1}^{p_2} \frac{dp}{\rho(p)}$

In case of inviscid, steady flow of incompressible fluid ($\rho = \text{const.}$), if $\underline{g}$ = - gradU and integration along streamlines:

$$\boxed{\frac{v_1^2}{2} + \frac{p_1}{\rho} + U_1 = \frac{v_2^2}{2} + \frac{p_2}{\rho} + U_2}$$

The Bernoulli's sum = const. along streamlines.

12. Static, dynamic and total pressure



$$\frac{v_1^2}{2} + \frac{p_1}{\rho} + U_1 = \frac{v_2^2}{2} + \frac{p_2}{\rho} + U_2$$

In stagnation point $\underline{v}$ = 0, so $p_\infty + \frac{\rho}{2} v_\infty^2 = p_t$

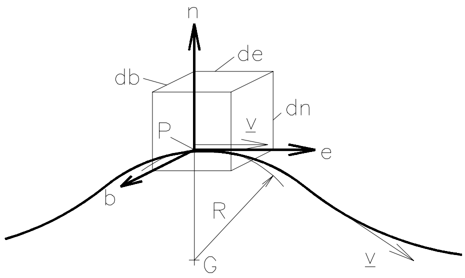
$p_d = \frac{\rho}{2} v_\infty^2$ dynamic pressure

p_∞ static pressure

p_t total, stagnation pressure

Bernoulli equation in case of inviscid, steady flow of incompressible fluid, disregarding the field of force: the total pressure is constant along streamlines.

13. Euler equation in streamwise ("natural") co-ordinate-system



Steady flow of inviscid ($\mu = 0$) fluid. **e** coordinate is tangent to the streamline, **n** is normal to it and cross the center of curvature, A **b** binormal coordinate perpendicular to **e** and **n**.

<u>In **e** direction</u>

Force acting on differential fluid particle of edge length db, dn and de (mass: $dm = \rho dbdnde$) in e direction:

$$dF_e = p\ db\ dn - \left[p + \left(\frac{\partial p}{\partial e} \right) de \right] db\ dn + \rho\ db\ dn\ de\ g_e,$$

where g_e the e component of the field of force.

Since the flow is steady only convective acceleration exists, and $v_n = v_b = 0$ $a_{conv} = v\frac{\partial v}{\partial e}$.

$$\rho dbdndev\frac{\partial v}{\partial e} = -\frac{\partial p}{\partial e} de\ db\ dn + \rho\ db\ dn\ de\ g_e.$$

$$\boxed{v\frac{\partial v}{\partial e} = -\frac{1}{\rho}\frac{\partial p}{\partial e} + g_e}$$

<u>In **n** direction</u>

$dm\frac{v^2}{R}$ centripetal force is needed to move dm mass with v velocity along a streamline of a radius of curvature R:

$$-\rho dedbdn\frac{v^2}{R} = pdbde - \left[p + \left(\frac{\partial p}{\partial n} \right) dn \right] dbde + \rho dedbdn g_n$$

$$\boxed{-\frac{v^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g_n}$$

<u>In **b** direction</u>

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial b} + g_b$$

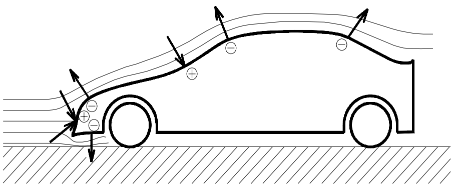
In normal co-ordinate direction, disregarding g:

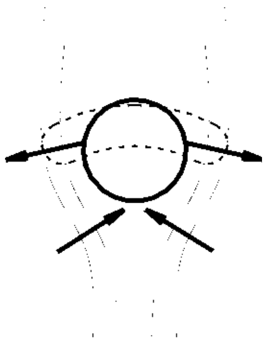
$$\boxed{\frac{v^2}{R} = \frac{1}{\rho}\frac{\partial p}{\partial n}}$$

Consequences:

a) if the streamlines are parallel straight lines (R=∞) the pressure doesn't change perpendicular to the streamlines,

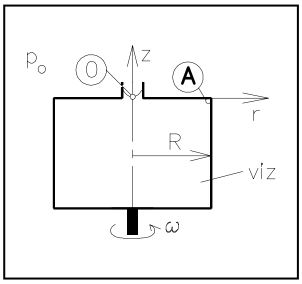
b) if the streamlines are curved the pressure changes perpendicular to the streamlines: it increases outwards from the center of curvature.





raindrop

14. Rotating tank



Forced vortex in absolute system, $\omega[1/s]$ angular velocity, $p_A - p_0 = ?$
3 different ways of solution:

a) co-rotating co-ordinate system: hydrostatics
b) absolute system Bernoulli equation;
c) absolute system, Euler equation in streamwise ("natural") co-ordinate-system

a) $p_A - p_0 = -\rho(U_A - U_0) = -\rho\left(-\frac{R^2\omega^2}{2} - 0\right) = \rho\frac{R^2\omega^2}{2}$

b) $\int_0^A \frac{\partial \underline{v}}{\partial t} d\underline{s} + \int_0^A \text{grad}\frac{v^2}{2} d\underline{s} - \int_0^A \underline{v} \times \text{rot}\underline{v} d\underline{s} = \int_0^A \underline{g} d\underline{s} - \int_0^A \frac{1}{\rho} \text{grad} p d\underline{s}$.

I II III IV V

Steady flow, integral I =0, integral II = $\left(v_A^2 - v_0^2\right)/2$, integral III $\neq 0$ since $\text{rot}\underline{v} \neq 0$, and no streamline connects points O and A. Since $\underline{g} \perp d\underline{s}$ integral IV =0, ρ = const. integral $V = -\left(p_A - p_0\right)/\rho$ $p_A - p_0 = \rho\int_0^A \underline{v} \times \text{rot}\underline{v} d\underline{s} - \rho\frac{v_A^2 - v_0^2}{2}$. $v = \omega r$ and $\left(\text{rot}\underline{v}\right)_z = dv/dr + v/r$ $\Rightarrow \left(\text{rot}\underline{v}\right)_z = 2\omega$. $\underline{v}$, $\text{rot}\underline{v}$ and $d\underline{s}$ vectors are perpendicular to each-other, and $|d\underline{s}| = dr$, furthermore $v_A = R\omega$ and $v_0 = 0$:

$$p_A - p_0 = \rho\int_0^R (r\omega)2\omega dr - \rho\frac{R^2\omega^2}{2} = \rho R^2\omega^2 - \rho\frac{R^2\omega^2}{2} = \rho\frac{R^2\omega^2}{2}.$$

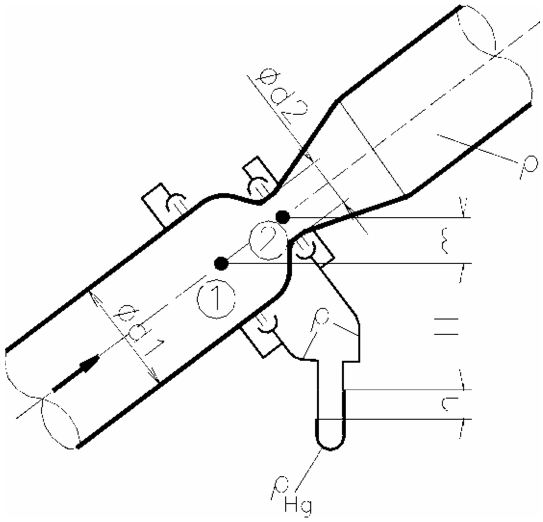
c)

$$\frac{v^2}{R} = \frac{1}{\rho}\frac{\partial p}{\partial n} - g_n.$$

R = r, dn = dr (streamlines are concentric circles), $g_n = 0$

$$\int_{p_0}^{p_A} dp = \int_0^R \rho\frac{v^2}{r} dr = \int_0^R \rho r\omega^2 dr \Rightarrow p_A - p_0 = \rho\frac{R^2\omega^2}{2}.$$

15. Measurement of flow rate by using Venturi meter



h [m]= f(q_v) = ? ρ and ρ_M density of water and mercury

$$\frac{v_1^2}{2}+\frac{p_1}{\rho}+U_1=\frac{v_2^2}{2}+\frac{p_2}{\rho}+U_2$$

U-tube manometer: $p_1+\rho g(H+h)=p_2+\rho g(m+H)+\rho_{Hg}gh \quad p_1-p_2=(\rho_{Hg}-\rho)gh+\rho gm$.

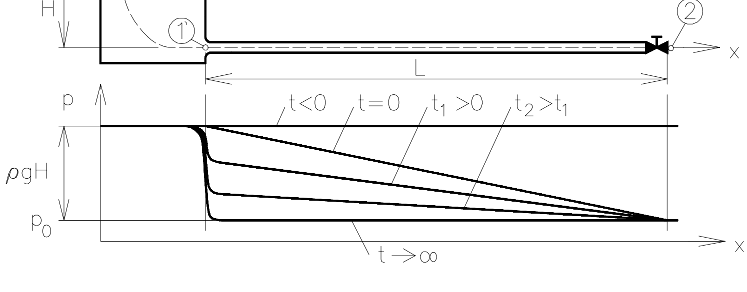
$$\frac{v_2^2}{2}-\frac{v_1^2}{2}=\frac{v_1^2}{2}\left[\left(\frac{v_2^2}{v_1^2}\right)^2-1\right]=\frac{p_1-p_2}{\rho}-gm=\frac{\rho_{Hg}-\rho}{\rho}gh$$

continuity equation: $v_1A_1 = v_2A_2 \Rightarrow v_2 / v_1 = (d_1 / d_2)^2$

$$v_1=\sqrt{\frac{\left(\frac{\rho_{Hg}}{\rho}-1\right)2gh}{\left(\frac{d_1}{d_2}\right)^4-1}}$$

Flow rate: $\boxed{q_v=\frac{d_1^2\pi}{4}v_1=K\sqrt{h}}$

16. Unsteady discharge of water from a tank



$$\int_1^2 \frac{\partial \underline{v}}{\partial t} d\underline{s} + \int_1^2 \text{grad}\frac{v^2}{2} d\underline{s} - \int_1^2 \underline{v} \times \text{rot}\underline{v} d\underline{s} = \int_1^2 \underline{g} d\underline{s} - \int_1^2 \frac{1}{\rho} \text{grad}p\, d\underline{s}$$

I II III IV V

$$\int_1^2 \frac{\partial \underline{v}}{\partial t} d\underline{s} + \left[\frac{v^2}{2} + \frac{p}{\rho} + U\right]_1^2 = 0$$

In point 1 $p_1 = p_0$, $z = H$, v = 0. In point 2 $p_2 = p_0$. $z = 0$, the velocity is $v_2 = v(t)$.

$$\int_1^2 \frac{\partial \underline{v}}{\partial t} d\underline{s} = \int_1^{1'} \frac{\partial \underline{v}}{\partial t} d\underline{s} + \int_{1'}^2 \frac{\partial \underline{v}}{\partial t} d\underline{s}.$$

where $\partial \underline{v} / \partial t$ acceleration vector $|\partial \underline{v} / \partial t|$ is indicated by a, $\partial \underline{v} / \partial t \,||\, d\underline{s}$, $\partial \underline{v} / \partial t$ and $d\underline{s}$ point at the same direction.

$$v_1 A_1 = v_2 A_2 \Rightarrow a_1 A_1 = a_2 A_2$$

$$\int_1^2 \frac{\partial \underline{v}}{\partial t} d\underline{s} = \int_{1'}^2 a ds = aL$$

$$\frac{dv}{dt} L + \frac{v^2}{2} + \frac{p_0}{\rho} = \frac{p_0}{\rho} + gH$$

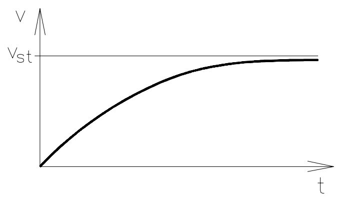
In case of steady flow $(\frac{dv}{dt} = 0)$ $\frac{v_{st}^2}{2} = gH$

$$\frac{dv}{v_{st}^2 - v^2} = \frac{dt}{2L}.$$

$$\int_0^{v/v_{st}} \frac{d\frac{v}{v_{st}}}{1 - \left(\frac{v}{v_{st}}\right)^2} = \frac{v_{st}}{2L} \int_0^t dt.$$

$$\text{arth}\frac{v}{v_{st}} = \frac{t v_{st}}{2L} \quad \tau = \frac{2L}{v_{st}}$$

$\frac{v}{v_{st}} = \text{th}\frac{t}{\tau}$ where $v_{st} = \sqrt{2gH}$.

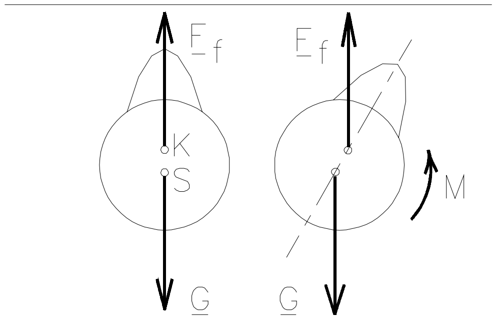


17. Floating of bodies

Body volume: ΔV, pressure distribution is characterized by $\text{grad}p$, Pressure force: $\Delta \underline{F} \cong -\text{grad}\, p\, \Delta V$ $\Delta \underline{F} = -\rho\, \underline{g}\, \Delta V$. In gravitational field buoyant force = weight of the volume displacement. The buoyant force vector crosses the center of displaced volume.

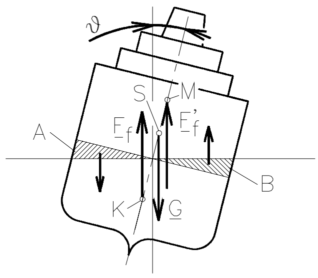
The body is floating if the average density is equal or less than the density of fluid.

Stability of floating body: submarines and ships.



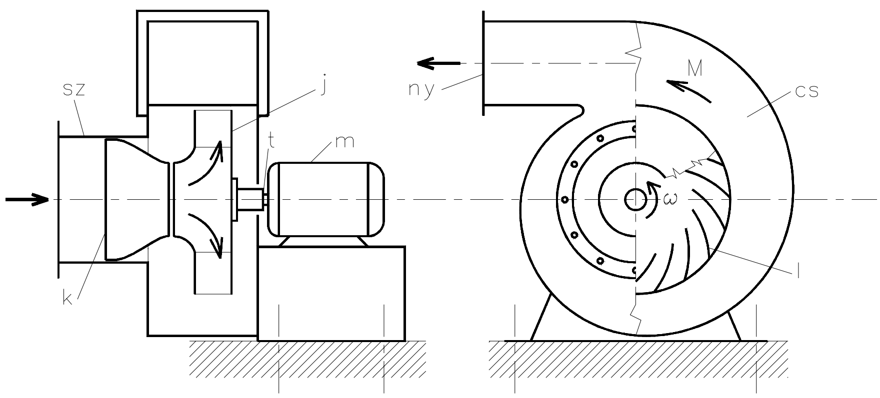
If the center of gravity S is lower than the center of displaced volume K, a moment M arises, decreasing the angle of deflection.

If S center of gravity is above the center of displaced volume K a moment is arising to a certain angle of deflection decreasing the angle of deflection.



At deflection the position, magnitude of weight and magnitude of buoyant force does not change. The line of application of buoyant force displaces. As a consequence of the deflection a wedge-shaped part of the body (A) emerges from the water and the B part of body sinks. So a couple of forces arise, displacing the buoyant force vector. The new line of application crosses the symmetry plane in point M (metacenter). If S is under metacenter M the ship is in stable equilibrium state.

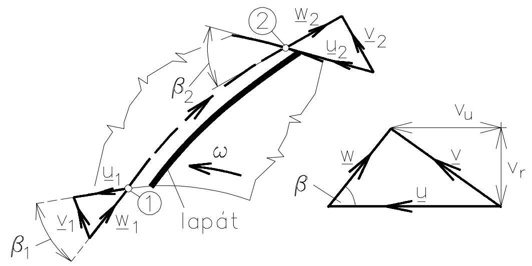
18. Radial-flow fan, Euler equation for turbines



sz: inlet, k: suction nozzle, j: impeller, l: blades, cs: casing, ny: outlet, t: shaft, m: electric motor, M: moment, ω: angular velocity.
Task: increase of total pressure of gas:

$$\Delta p_t = p_{ot} - p_{it} = \left(p + \frac{\rho}{2}v^2\right)_o - \left(p + \frac{\rho}{2}v^2\right)_i$$

available performance: $\boxed{P = q_v \Delta p_t}$, where q_v [kg/m^3] is the flow rate.
Bernoulli equation in relative coordinate-system (steady flow of incompressible and inviscid fluid) between points 1 and 2 of the same streamline.:



$$\int_1^2 \frac{\partial \underline{w}}{\partial t} d\underline{s} + \frac{w_2^2 - w_1^2}{2} - \int_1^2 \underline{w} \times \mathrm{rot}\underline{w} d\underline{s} = \int_1^2 \underline{g} d\underline{s} - \int_1^2 \frac{1}{\rho} \mathrm{grad} p \, d\underline{s} .$$

I II III IV V

$$\underline{g} = \underline{g}_c + \underline{g}_{Cor} = -\mathrm{grad}(U_g + U_c) + 2\underline{w} \times \underline{\omega}, \; U_g \cong 0, \; U_c = -\frac{r^2\omega^2}{2}$$

$$\int_1^2 \underline{g} d\underline{s} = \left(\frac{r_2^2\omega^2}{2} - \frac{r_1^2\omega^2}{2}\right) + \int_1^2 2\underline{w} \times \underline{\omega} d\underline{s},$$

Since $\underline{v} = \underline{w} + \underline{u}$, if $\mathrm{rot}\underline{v} = \underline{0} \Rightarrow \mathrm{rot}\underline{w} = -\mathrm{rot}\underline{u}$. Since $|\underline{u}| = r\omega$ $\mathrm{rot}\underline{u} = 2\underline{\omega}$.

$$-\int_1^2 \underline{w} \times \mathrm{rot}\underline{w} d\underline{s} = -\int_1^2 \underline{w} \times (-2\underline{\omega}) d\underline{s} = \int_1^2 2\underline{w} \times \underline{\omega} d\underline{s}$$

Finally:

$$\frac{w_1^2}{2} + \frac{p_1}{\rho} - \frac{r_1^2\omega^2}{2} = \frac{w_2^2}{2} + \frac{p_2}{\rho} - \frac{r_2^2\omega^2}{2}$$

$$\underline{w} = \underline{v} - \underline{u} \Rightarrow w^2 = v^2 + u^2 - 2\underline{u}\underline{v}$$

$$\frac{v_2^2}{2} + \frac{u_2^2}{2} - \underline{v}_2\underline{u}_2 - \frac{r_2^2\omega^2}{2} - \frac{v_1^2}{2} - \frac{u_1^2}{2} + \underline{v}_1\underline{u}_1 + \frac{r_1^2\omega^2}{2} + \frac{p_2 - p_1}{\rho} = 0.$$

$u_1 = r_1\omega$

$$\Delta p_t = p_{2t} - p_{1t} = \left(p_2 + \frac{\rho}{2}v_2^2\right) - \left(p_1 + \frac{\rho}{2}v_1^2\right) = \rho(\underline{v}_2\underline{u}_2 - \underline{v}_1\underline{u}_1).$$

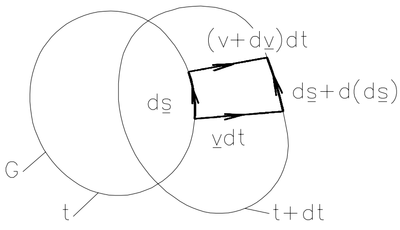
$$\underline{v}_2\underline{u}_2 = v_{2u}u_2,$$

$$\boxed{\Delta p_{t\,id} = \rho(v_{2u}u_2 - v_{1u}u_1)}$$

If $\underline{v}_{1u} = 0 \Rightarrow \quad \Delta p_{t\,id} = \rho v_{2u} u_2 .$

19. Theorems for vorticity: Thomson' and Helmholtz' theorems

Thomson' theorem (inviscid fluid)

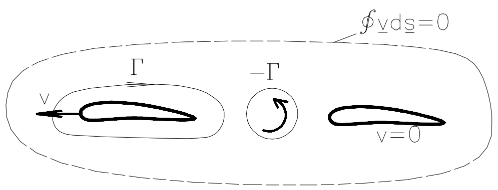


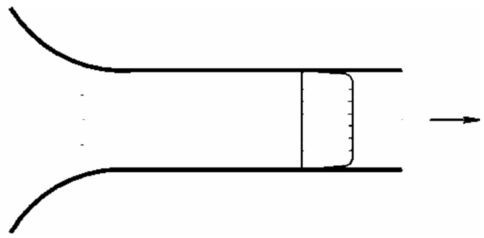
Circulation: $\Gamma = \oint_G \underline{v}\, d\underline{s}$. Temporal change of circulation along closed fluid line $\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_G \underline{v}\, d\underline{s} = ?$ If $\underline{g} = -\text{grad}U$ and ρ=const. or ρ=ρ(p), by using Euler equation:

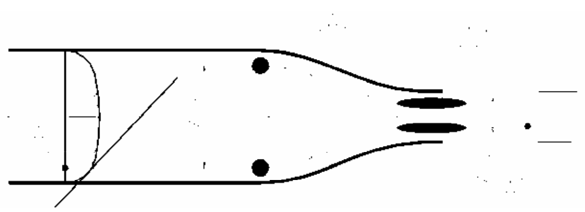
$$\boxed{\frac{d}{dt}\oint_G \underline{v} d\underline{s} = 0}$$

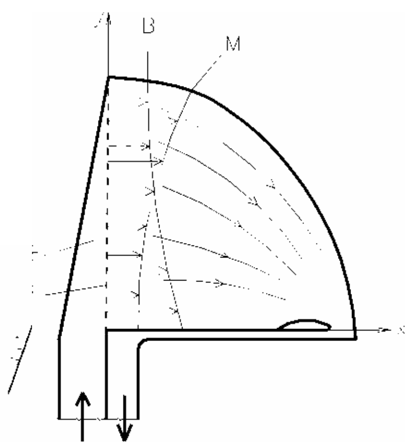
In flow of incompressible and inviscid fluid in potential field of force no vorticity arises.

Applications:
Starting and stopping vortex (vortex shedding), making velocity distribution uniform, flow in water reservoir





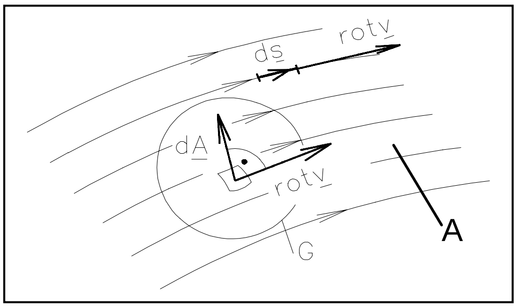




$\Gamma = \oint_G \underline{v} d\underline{s} = \int_A \text{rot}\underline{v} d\underline{A}$. $\frac{(\text{rot}\,\underline{v})_{\vartheta 2}}{(\text{rot}\,\underline{v})_{\vartheta 1}} = \frac{\Delta A_1}{\Delta A_2} = \frac{D_2}{D_1}$. $(\text{rot}\,\underline{v})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$.$\partial v_y/\partial x<0 \Rightarrow \partial v_x/\partial y<0$.

Helmholtz' I. theorem $\boxed{\mu = 0}$

Fluid vortex line: rot $\underline{v} \times d\underline{s} = 0$, fluid vortex surface: rot $\underline{v} \times d\underline{A} = 0$

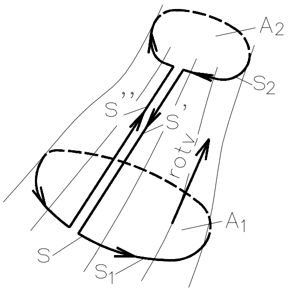


Since $\frac{d}{dt}\oint_G \underline{v}d\underline{s} = 0$, a flowing vortex surface remains vortex surface.
Two vortex sheets intersect each other along a vortex line.

A flowing vortex line, which can be regarded as line of intersection of two flowing vortex surfaces, consists of the same fluid particles.

Consequence: The vortex in smoke ring or in cloud of smoke emerging from a chimney preserves the smoke.

Helmholtz' II. theorem



Flowing vortex tube

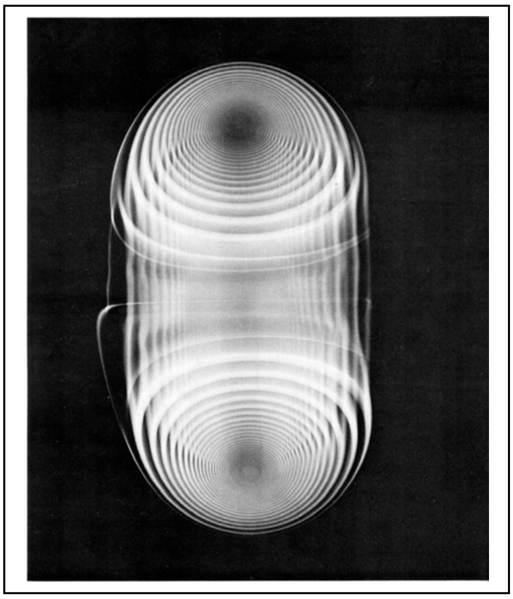
$$\oint_S \underline{v}d\underline{s} = \oint_{S_1} \underline{v}d\underline{s} + \oint_{S_2} \underline{v}d\underline{s} = 0$$

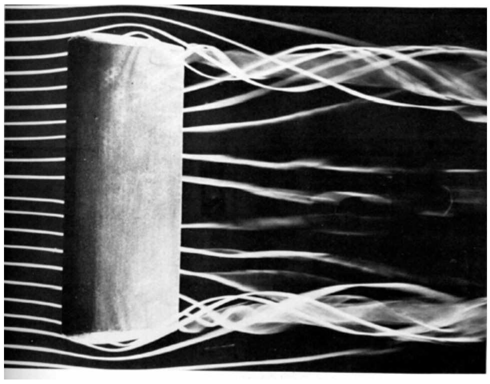
$$\oint_{S_1} \underline{v}d\underline{s} = \oint_{S_2} \underline{v}d\underline{s} \, ,$$

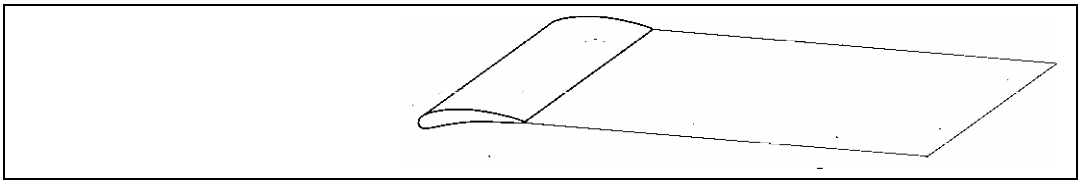
$$\int_{A_1} rot\underline{v}d\underline{A} = \int_{A_2} rot\underline{v}d\underline{A} \, .$$

$\int_A rot\underline{v}d\underline{A}$ is constant over all cross sections along a vortex tube and it does not change temporally.

Consequences: the vortex tube is either a closed line (a ring) or ends at the boundary of the flow field. A $\Rightarrow$ 0 rot$\underline{v}$ $\Rightarrow \infty$.







Induced vortex, tip vortex of finite airfoil.

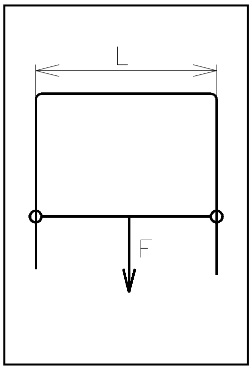
Flight of wild-gooses in V shape.

Vortex in tub after opening the sink.

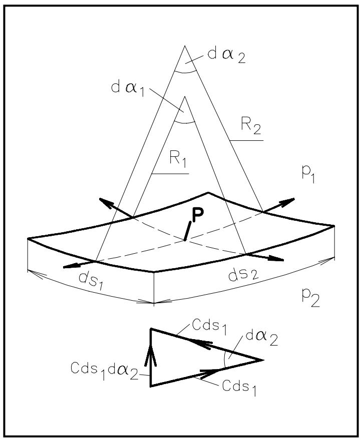
Tornado

20.Pressure measurements

Surface tension



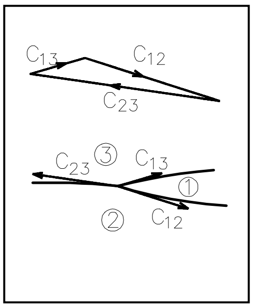
$F = 2LC$ C [N/m] surface tension coefficient. For water air combination $C = 0.025[N/m]$.



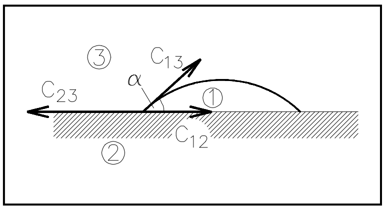
$(p_1 - p_2)ds_1 ds_2 = Cds_1 d\alpha_2 + Cds_2 d\alpha_1$.

$$\Delta p = p_1 - p_2 = C\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

In case of spheres $R_1 = R_2 = R \Rightarrow \Delta p = 2C/R$, and bubbles: $\Delta p = 4C/R$



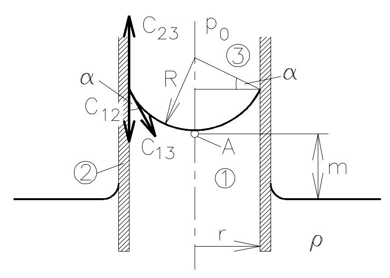
If $|C_{23}| > |C_{12}| + |C_{13}|$, fluid 1 expands on the surface of fluid 2 (e.g. oil on water.



$C_{23} ds = C_{12} ds + C_{13} \cos\alpha ds$.

$\cos\alpha = (C_{23} - C_{12})/C_{13}$. $C_{23} > C_{12} \Rightarrow \alpha < 90°$, $\alpha > 90°$ (mercury) Ha $|C_{23}| > |C_{12}| + |C_{13}|$, the fluid expands over the surface of solid body petroleum gets out of open bottle.

Capillary rise



$$p_0 - p_A = 2C_{13}/R = 2C_{13}\cos\alpha / r\,.$$

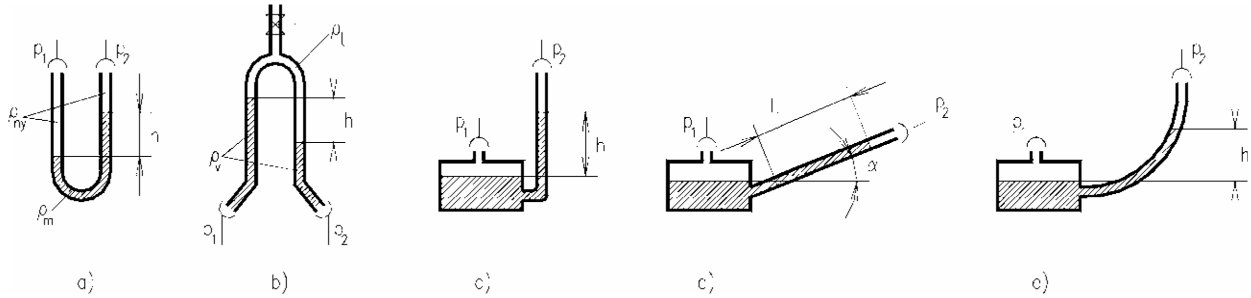
$$p_0 - p_A = \rho g m$$

$$m = \frac{2C_{13}}{\rho g r}\cos\alpha$$

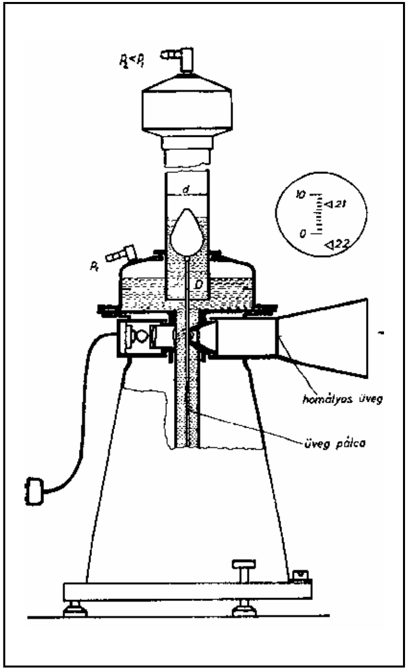
In case of mercury capillary drop.

Measurement of pressure

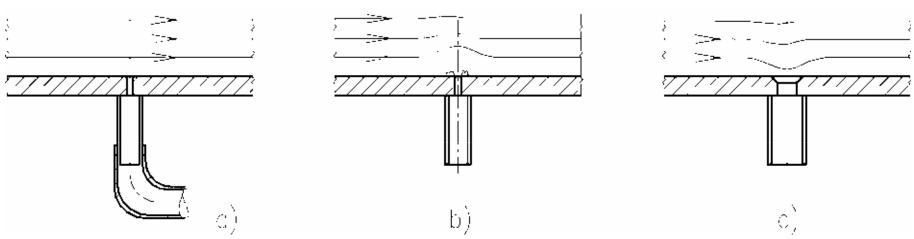
Manometers (for measuring pressure differences)
Micromanometers: U-tube manometer $p_1 - p_2 = (\rho_m - \rho_t)\, g\, h$, "inverse" U-tube manometer $p_1 - p_2 = (\rho_w - \rho_a)\, g\, h$, inclined tube manometer $L = H/\sin\alpha$, relative error: $e = \Delta s/L = (\Delta s/H)\sin\alpha$, bent tube manometer (e = const.):



Betz-micromanometer

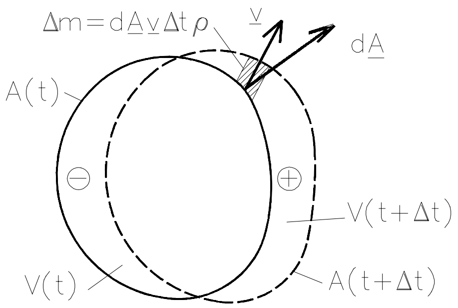


Pressure taps



21. Integral momentum equation

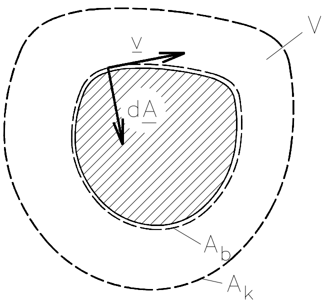
$$\frac{d}{dt}\int_{V(t)} \rho \underline{v} dV = \int_{V(t)} \rho \underline{g} dV - \int_{A(t)} p d\underline{A}$$ $\mu = 0$



$$\frac{d}{dt}\int_{V(t)} \rho \underline{v} dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} (\rho \underline{v})_{t+\Delta t} dV - \int_{V(t)} (\rho \underline{v})_t dV\right]$$

$$\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_A \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p d\underline{A}$$ where V is the control volume

Solid body in control volume



$$\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_{A_k} \underline{v}\rho(\underline{v} d\underline{A}) + \int_{A_b} \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_{A_k} p d\underline{A} - \int_{A_b} p d\underline{A}$$

$$\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_A \underline{v}\rho(\underline{v}\, d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p\, d\underline{A} - \underline{R}$$

$\underline{R}$[N] force acting on the body in the control volume

In case of steady flow:

$$\int_A \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p d\underline{A} - \underline{R}$$

$$\boxed{\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_A \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p d\underline{A} - \underline{R} + \underline{S}}$$

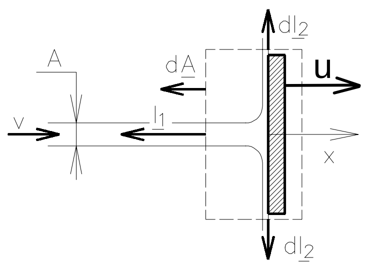
where $\underline{S}$ is the friction force.

Moment-of-momentum equation

$$\frac{\partial}{\partial t}\int_V \underline{r}\times(\rho\underline{v})dV + \int_A \underline{r}\times\underline{v}\rho(\underline{v}d\underline{A}) = \int_V \underline{r}\times\rho\underline{g}dV - \int_A \underline{r}\times pd\underline{A} - \underline{M} + \underline{M}_s$$

22. Application of integral momentum equation

Stationary and moving deflector



1. Taking up the control volume: solid body is in the control volume, surfaces are ⊥ or || to the velocity vector.

$$\frac{\partial}{\partial t}\int_V (\rho\underline{v})dV + \int_A \underline{v}\rho(\underline{v}d\underline{A}) = \int_V \rho\underline{g}dV - \int_A pd\underline{A} - \underline{R} + \underline{S}$$

I II III IV V VI

2. Simplification of equation

$$\int_A \underline{v}\rho(\underline{v}d\underline{A}) = -\underline{R}$$

3. Evaluation of integrals

$$\underline{I}_1 = \int_{A_1} \underline{v}\rho(\underline{v}d\underline{A}) = \rho v_1^2 A_1\left(-\underline{v}_1/|\underline{v}|_1\right),$$

$$d\underline{I}_2 = \rho v_2^2 dA_2 \frac{\underline{v}_2}{|\underline{v}|_2}$$

$$|\underline{I}| = \rho v^2 A\,.$$

4. Balances of forces in co-ordinate directions

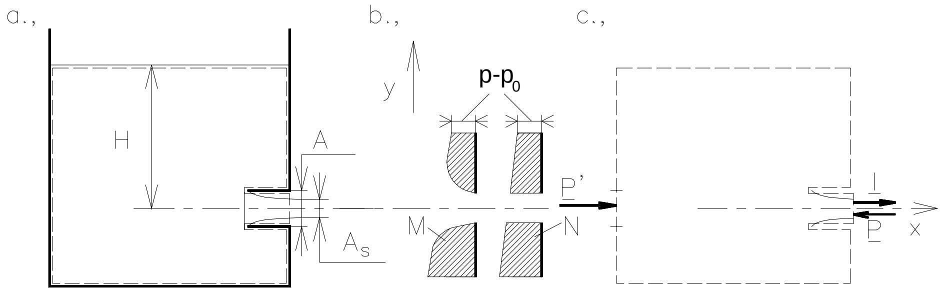
$$-I_1 = -R_x \text{ azaz} - \rho v_1^2 A_1 = -R_x = -R$$

$$R = \rho v_1^2 A_1$$

If the deflector moves: $\underline{u} > 0$

instead of $\underline{v}_1$ $\underline{w}_1 = \underline{v}_1 - \underline{u} \Rightarrow R = \rho(v_1 - u)^2 A_1$.

Borda discharge nozzle, contraction of jet

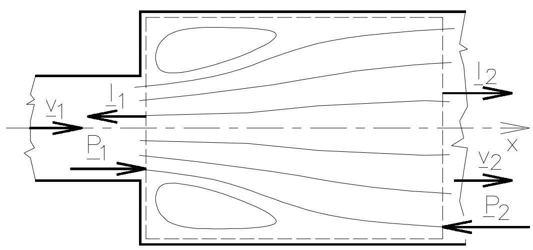


$$\alpha = A_S / A \qquad v = \sqrt{2gH}$$

$$I = P' - P \;\Rightarrow\; \rho v^2 A_s = \left(p_0 + \rho g H\right)A - p_0 A$$

$$2\rho g H A_s = \rho g H A \;\Rightarrow\; \alpha = A_s / A = 0.5$$

Change of pressure in Borda-Carnot sudden enlargement



$$-I_1 + I_2 = P_1 - P_2\,, \quad -\rho v_1^2 A_1 + \rho v_2^2 A_2 = -p_2 A_2 + p_1 A_2\,.$$

$$\rho v_1 A_1 = \rho v_2 A_2\,, \quad \left(p_2 - p_1\right)_{BC} = \rho v_2\left(v_1 - v_2\right), \quad \left(p_2 - p_1\right)_{id} = \frac{\rho}{2}\left(v_1^2 - v_2^2\right)$$

$$\Delta p'_{BC} = \left(p_2 - p_1\right)_{id} - \left(p_2 - p_1\right)_{BC} = \frac{\rho}{2}\left(v_1^2 - v_2^2\right) - \rho v_2\left(v_1 - v_2\right)$$

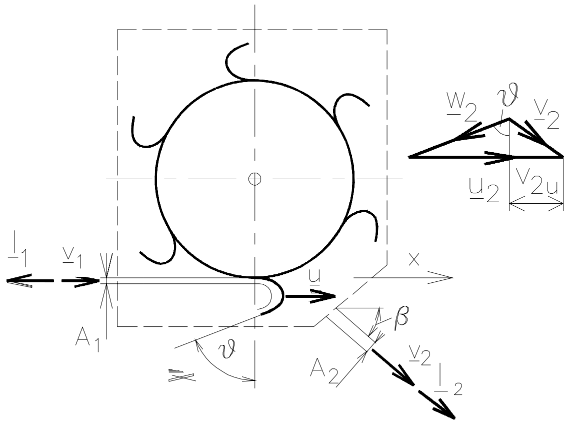
Borda-Carnot loss

$$\Delta p'_{BC} = \frac{\rho}{2}\left(v_1 - v_2\right)^2$$

Force acting on Borda-Carnot sudden enlargement

$$\underline{R} = \left(p_0 - p_1\right)\left(A_2 - A_1\right) = \rho v_2\left(v_1 - v_2\right)\left(A_2 - A_1\right)$$

A Pelton-wheel



$$-I_1 + I_{2u} = -R_u\,,\quad I_1 = \rho v_1^2 A_1, I_{2u} = \rho v_2^2 A_2 \cos\beta\,,\ \rho v_1 A_1 = \rho v_2 A_2$$

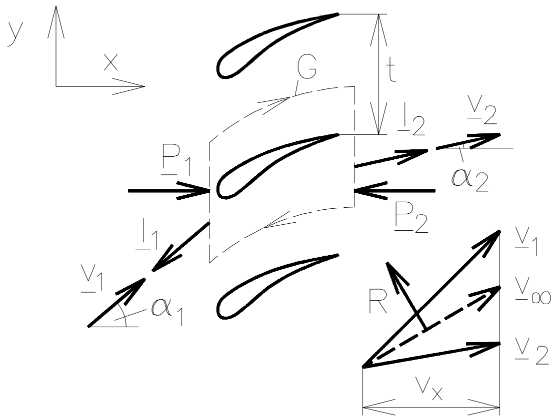
$$R_u = \rho v_1 A_1 (v_1 - v_{2u})\,,\ \text{where}\ v_{2u} = v_2 \cos\beta$$

$$v_{2u} = u - w_2 \sin\vartheta\,,\ w_2 = w_1\,,\ w_1 = v_1 - u\,,\quad R_u = \rho v_1 A_1 (v_1 - u)(1 + \sin\vartheta)$$

$$R_u = \rho v_1 A_1 (v_1 - u)(1 + \sin\vartheta)\,,\ \vartheta = 90^\circ\ \frac{\partial P}{\partial u} = 2\rho v_1 A_1 [(v_1 - u) - u] = 0\,,\ u = v_1/2$$

$$P_{max} = \rho v_1 A_1 v_1^2 / 2$$

Force acting on one element of a infinite blade row, Kutta-Joukowsky theorem



$$-I_{1x} + I_{2x} = P_1 - P_2 - R_x\,,\quad -I_{1y} + I_{2y} = -R_y$$

$$I_1 = \rho v_x t v_1\,,\ I_2 = \rho v_x t v_2\,,\ P_1 = p_1 t \quad P_2 = p_2 t\,,$$

$$R_x = (p_1 - p_2)t + \rho v_x t (v_1 \cos\alpha_1 - v_2 \cos\alpha_2)$$

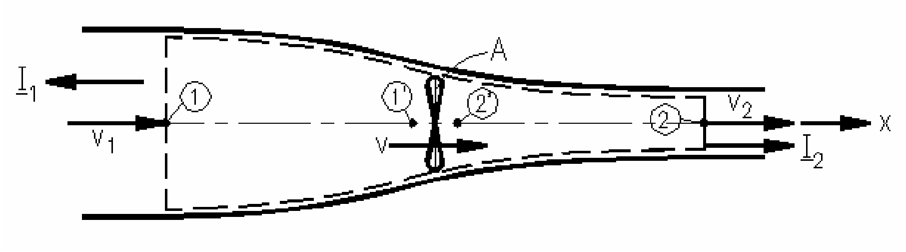
$$v_{1x} = v_{2x} \Rightarrow p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2) = \frac{\rho}{2}(v_{2y}^2 - v_{1y}^2) \Rightarrow R_x = \frac{\rho}{2}(v_{2y} - v_{1y})t(v_{2y} + v_{1y})$$

$$R_y = \rho v_x t (v_{1y} - v_{2y})\,.\ \Gamma = (v_{1y} - v_{2y})t \Rightarrow R_x = -\rho\Gamma \frac{v_{1y} + v_{2y}}{2}\,, R_y = \rho \Gamma v_x\,,$$

$$|\underline{R}| = \sqrt{R_x^2 + R_y^2} = \rho\Gamma \sqrt{v_x^2 + \left(\frac{v_{1y} + v_{2y}}{2}\right)^2} \Rightarrow |\underline{R}| = \rho |\underline{v}_\infty| \Gamma\ [N/m]$$

$$t \to \infty\,,\ v_{1y} - v_{2y} \to 0\ \ \Gamma = t(v_{1y} - v_{2y}) = \text{áll.}\ \ \underline{v}_2 \to \underline{v}_1 \to \underline{v}_\infty\ \ |\underline{R}| = \rho |\underline{v}_\infty| \Gamma\ [N/m]$$

Propellers (actuator disc)



v_1 = u, velocity $v_1 \Rightarrow v_2$

$$\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_A \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p d\underline{A} - \underline{R} + \underline{S}$$

I II III IV V VI

$$-I_1 + I_2 = -R_x \Rightarrow R_x = \rho v_1^2 A_1 - \rho v_2^2 A_2$$

By using continuity equation: $R_x = \rho v A(v_1 - v_2)$

Bernoulli-equations 1 - 1' and 2'- 2

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} = \frac{v_{1'}^2}{2} + \frac{p_{1'}}{\rho} \qquad \frac{v_{2'}^2}{2} + \frac{p_{2'}}{\rho} = \frac{v_2^2}{2} + \frac{p_2}{\rho}$$

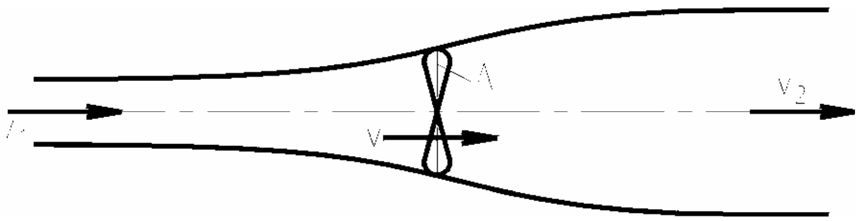
Since $p_1 = p_2$ and $v_{1'} = v_{2'} = v$, $p_{1'} - p_{2'} = \frac{\rho}{2}\left(v_1^2 - v_2^2\right)$

$$R_x = \frac{\rho}{2}(v_1^2 - v_2^2)A = \rho v A(v_1 - v_2), \; v = \frac{v_1 + v_2}{2} \Rightarrow R_x = \frac{\rho}{2}(v_1^2 - v_2^2)A$$

$P_u = v_1 R$ (useful power) $P_i = v\,R$ (input power) theoretical propellor efficiency:

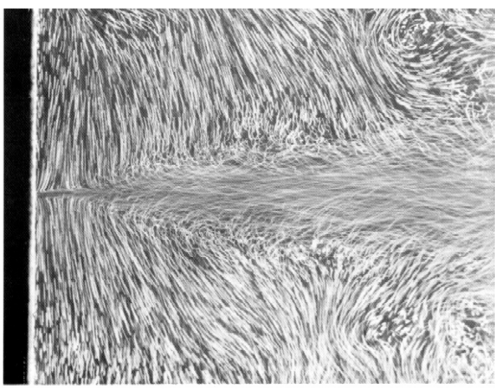
$$\eta_p = \frac{v_1 R}{vR} = \frac{v_1}{v} = \frac{2}{1 + v_2/v_1}$$

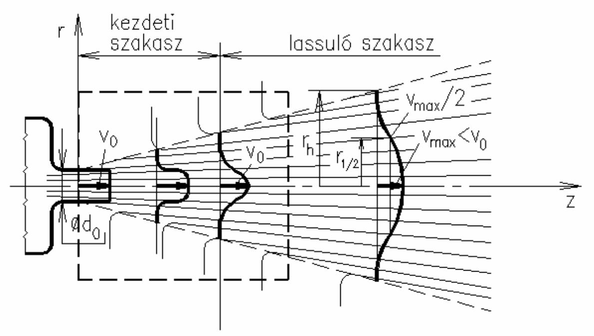
Wind turbine



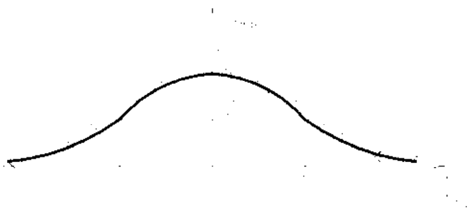
$$P = \rho A \frac{v_1 + v_2}{2}(v_1^2 - v_2^2) \Rightarrow \; \partial P/\partial v_2 = 0 \Rightarrow v_2 = \frac{1}{3}v_1 \Rightarrow \; P = \frac{16}{27}\rho A v_1^3$$

Jet





kezd.sz.:initial section, lassuló sz.: deceleration section



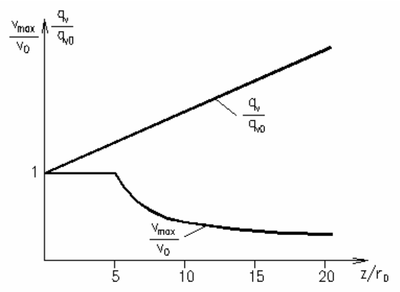
$r_h \sim z$, $r_{1/2} = k_1 z$,

$r_{1/2}/r_0 = k_1 z/r_0$

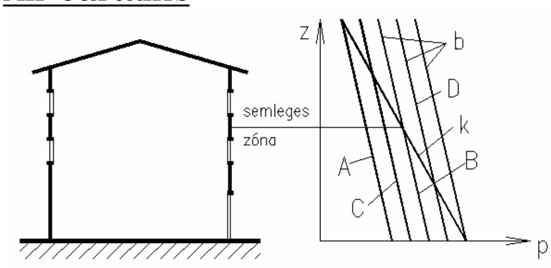
$$\int_A \underline{v}\rho\left(\underline{v}\,d\underline{A}\right) = 0 \quad \Rightarrow \rho v_0^2 r_0^2 \pi = \int_A \rho\, v^2\, 2\, r\, \pi\, dr \Rightarrow v_0^2 r_0^2 = v_{max}^2 r_{1/2}^2\, 2 \int_0^{r_h/r_{1/2}} \frac{v^2}{v_{max}^2} \frac{r}{r_{1/2}}\, d\frac{r}{r_{1/2}}$$

$$\frac{v_{max}}{v_0} = \frac{\text{Konst.}}{\frac{r_{1/2}}{r_0}} \Rightarrow \frac{v_{max}}{v_0} = \frac{\text{Konst.}}{\frac{z}{r_0}}$$

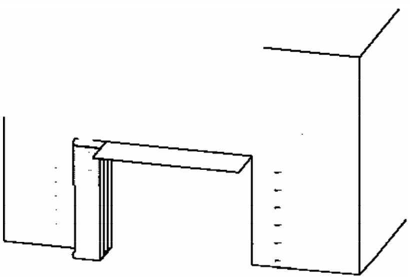
$$q_v = \int_0^{r_h} 2r\pi v\, dr = v_{max} r_{1/2}^2\, 2\pi \int_0^{r_h/r_{1/2}} \frac{v}{v_{max}} \frac{r}{r_{1/2}}\, d\frac{r}{r_{1/2}} \Rightarrow \frac{q_v}{q_{v0}} = \text{Konst}\, \frac{v_{max}}{v_0} \frac{r_{1/2}^2}{r_0^2} \quad \Rightarrow \frac{q_v}{q_{v0}} = \text{Konst.} \frac{z}{r_0}$$

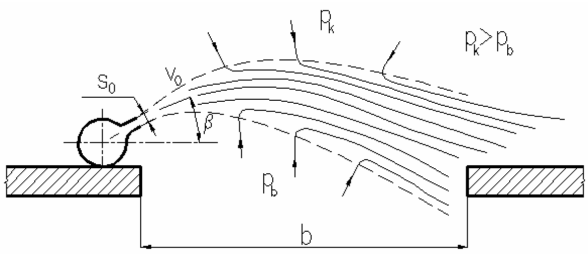


Air curtains

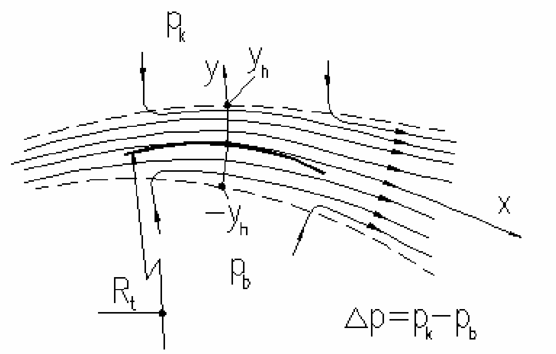


Neutral height





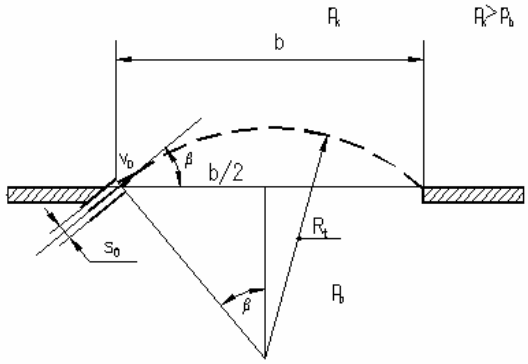
a.



b.

$$\frac{v^2}{R}=\frac{1}{\rho}\frac{\partial p}{\partial n}\Rightarrow\int_{p_b}^{p_k}dp=\int_{-y_h}^{y_h}\rho\frac{v^2}{R}dy\Rightarrow\ \Delta p=p_k-p_b=\frac{1}{R}\int_{-y_h}^{y_h}\rho v^2dy\Rightarrow\ \rho v_0^2 s_0=\int_{-y_h}^{y_h}\rho v^2dy$$

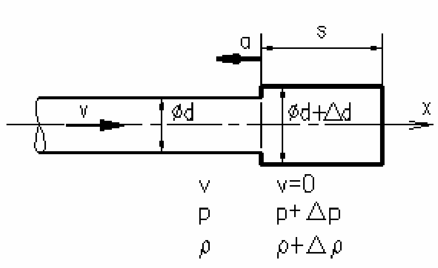
$$\Delta p=\frac{\rho v_0^2 s_0}{R}\Rightarrow R=\frac{\rho v_0^2 s_0}{\Delta p},$$

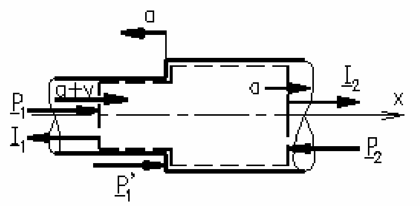


b=2 R_t sinβ. $b=2\frac{\rho v_0^2 s_0}{\Delta p}\sin\beta$, $B=\frac{b}{s_0}$, $D=\frac{\Delta p}{\frac{\rho}{2}v_0^2}$, $B=\frac{K}{D}\sin\beta$

theoretical: K=4, experimental: K=1.71+0.0264 B ($25\leq\beta^0\leq45$ and $10\leq B\leq40$)

<u>Allievi theorem</u>





pipe diameter d ⇒ d+Δd

Shortening of water column. Compression of water: Δs_1 and expansion of pipe wall: Δs_2 ⇒ $\Delta s = \Delta s_1 + \Delta s_2$

$\Delta s_1 = \frac{\Delta p}{E_v} s$, E_v = 2.1·10^{9}Pa, $\Delta s_2 \frac{d^2\pi}{4} = sd\pi \frac{\Delta d}{2}$ $\Delta\sigma = \frac{\Delta p\, d}{2\delta}$; $\frac{\Delta d}{d} = \frac{\Delta\sigma}{E_a} = \frac{\Delta p\, d}{2\delta\, E_a}$

E_{st} = 2·10^{11}Pa $\Rightarrow \Delta s_2 = \frac{\Delta p\, d}{\delta\, E_a} s \Rightarrow \frac{\Delta s}{s} = \frac{\Delta s_1}{s} + \frac{\Delta s_2}{s} = \Delta p\left(\frac{1}{E_v} + \frac{d}{\delta\, E_a}\right) = \frac{\Delta p}{E_r}$

Integral momentum equation:

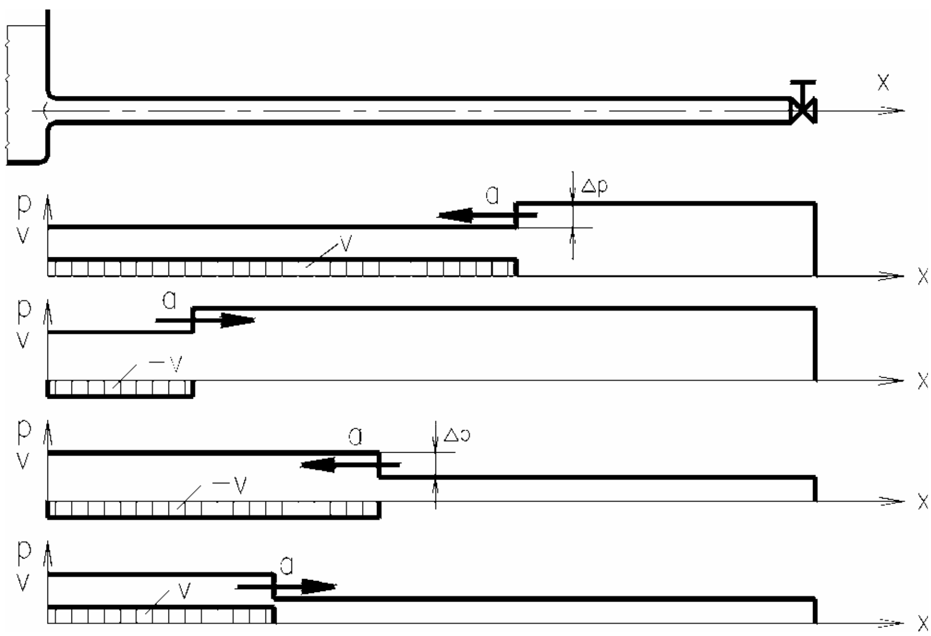
$$-\rho(a+v)A(a+v) + (\rho+\Delta\rho)\, a\, (A+\Delta A)a =$$

$$= pA + (p+\Delta p)\Delta A - (p+\Delta p)(A+\Delta A),$$

Continuity: $\rho(a+v)A = (\rho+\Delta\rho)a(A+\Delta A)$

$\Delta p = \rho v(a+v)$ v<<a $\Delta p = \rho v a$ $T_t \le \frac{2L}{a}$ turning off time

s = at $\Rightarrow A\,\Delta s = A\frac{\Delta p}{E_r} s = A\frac{\Delta p}{E_r} at \Rightarrow \frac{\Delta p}{E_r} a = v$ $a = \sqrt{\frac{E_r}{\rho}}$.



23. Flow of viscous fluids, Navier-Stokes equation

Rheology

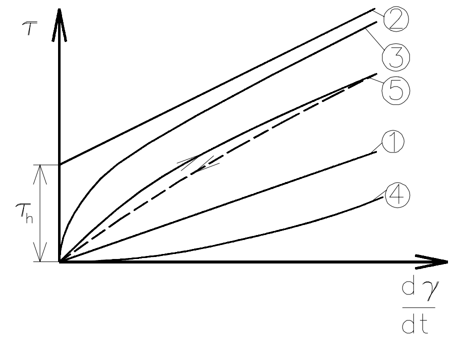
Shear stress versus strain rate (deformation rate)

curve 1: Newtonian fluid: $\underline{\tau_{yx} = \mu \frac{dv_x}{dy}} = \mu \frac{d\gamma}{dt}$.

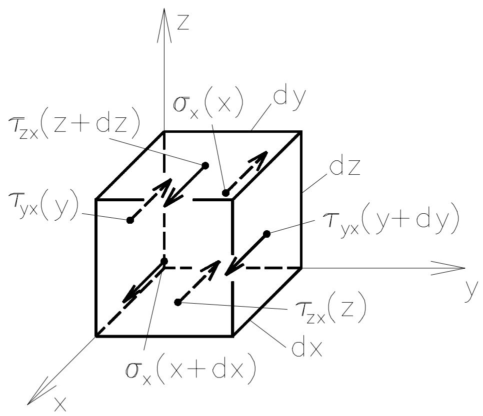
Non-Newtonian fluids:

curve 2: $\tau = \tau_h + \mu_\infty \frac{d\gamma}{dt}$,

Curves 3 and 4: $\tau = k(d\gamma / dt)^n$.



Differential momentum equation $\frac{d\underline{v}}{dt} = \underline{g} + \underline{F}$, In case of inviscid fluid: $\underline{F} = -\frac{1}{\rho}\text{grad}p$



$$F_x = \frac{1}{\rho dxdydz}\{[\sigma_x(x+dx) - \sigma_x(x)]dydz + [\tau_{yx}(y+dy) - \tau_{yx}(y)]dxdz$$
$$+ [\tau_{zx}(z+dz) - \tau_{zx}(z)]dxdy\}$$

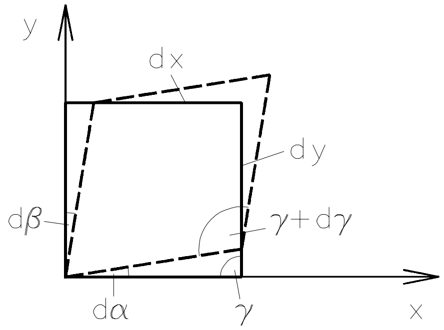
Since $\sigma_x(x+dx) = \sigma_x(x) + \frac{\partial \sigma_x}{\partial x}dx \Rightarrow F_x = \frac{1}{\rho}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$

$$\underline{\underline{\Phi}} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \qquad \underline{F} = \frac{1}{\rho}\underline{\underline{\Phi}}\underline{\nabla} \quad \Rightarrow \quad \underline{\nabla} = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$\boxed{\frac{d\underline{v}}{dt} = \underline{g} + \frac{1}{\rho}\underline{\underline{\Phi}}\underline{\nabla}}$$

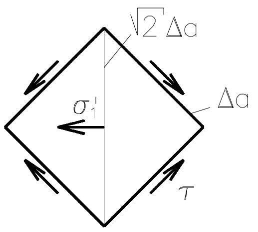
$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}=g_x+\frac{1}{\rho}\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)$$

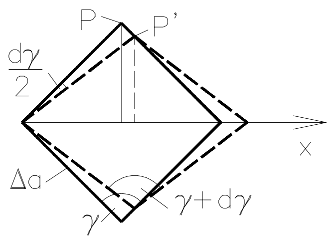
Relation between stresses and deformation.



$$d\gamma = d\alpha + d\beta = \frac{\partial v_y}{\partial x}dt+\frac{\partial v_x}{\partial y}dt\ .$$

$$\tau_{xy}=\mu\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)=\tau_{yx}\ . \qquad p=-\frac{1}{3}\left(\sigma_x+\sigma_y+\sigma_z\right).$$





$$\sigma'_{1x}=\mu\frac{d\gamma}{dt}=2\mu\frac{\partial v_x}{\partial x}$$

$$\sigma_x=-p+2\mu\frac{\partial v_x}{\partial x}-\frac{2}{3}\mu\,\mathrm{div}\underline{v}\ .$$

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left[2\mu\frac{\partial v_x}{\partial x}-\frac{2}{3}\mu\,\mathrm{div}\underline{v}\right]+\right.$$
$$\left.+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\right]\right\}.$$

Navier-Stokes-equation

$\mu=\text{const.}$ and $\rho=\text{const.}$

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$
$$\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}=g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)$$
$$\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}=g_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

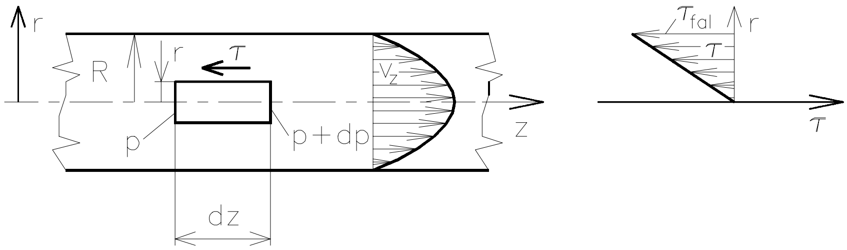
$$\boxed{\frac{d\underline{v}}{dt}=\frac{\partial \underline{v}}{\partial t}+\mathrm{grad}\frac{v^2}{2}-\underline{v}\times \mathrm{rot}\underline{v}=\underline{g}-\frac{1}{\rho}\mathrm{grad}p+\nu\Delta\underline{v}}$$ Navier-Stokes-equation

$\Delta\underline{v}=\mathrm{grad}\,\mathrm{div}\underline{v}-\mathrm{rot}\,\mathrm{rot}\underline{v}$.

Since $\mathrm{div}\underline{v}=0$ $\boxed{\frac{d\underline{v}}{dt}=\underline{g}-\frac{1}{\rho}\mathrm{grad}p-\nu\,\mathrm{rot}\,\mathrm{rot}\underline{v}}$ if rot rot $\underline{v}$ = 0, rot $\underline{v}$ = 0 viscosity exerts no effect.

24. Developed laminar pipe flow

Cylindrical symmetry of developed pipe flow: $v_r=0$, $\frac{\partial(\;)}{\partial z}=0$ (2D flow)



$r^2\pi p-r^2\pi(p+dp)+2r\pi dz\tau=0$. $\Rightarrow$ $2\tau dz=rdp$ $\Rightarrow$ $\tau=\frac{1}{2}r\frac{dp}{dz}=\mu\frac{dv_z}{dr}$

dp/dz= const.

$$\int dv_z=\frac{1}{2\mu}\frac{dp}{dz}\int rdr\Rightarrow v_z=\frac{1}{4\mu}\frac{dp}{dz}r^2+\text{áll.}$$

If r=R $\Rightarrow$ $v_z=0$ $v_z=-\frac{1}{4\mu}\frac{dp}{dz}\left[R^2-r^2\right]\Rightarrow v_z>0$ if $\frac{dp}{dz}<0$

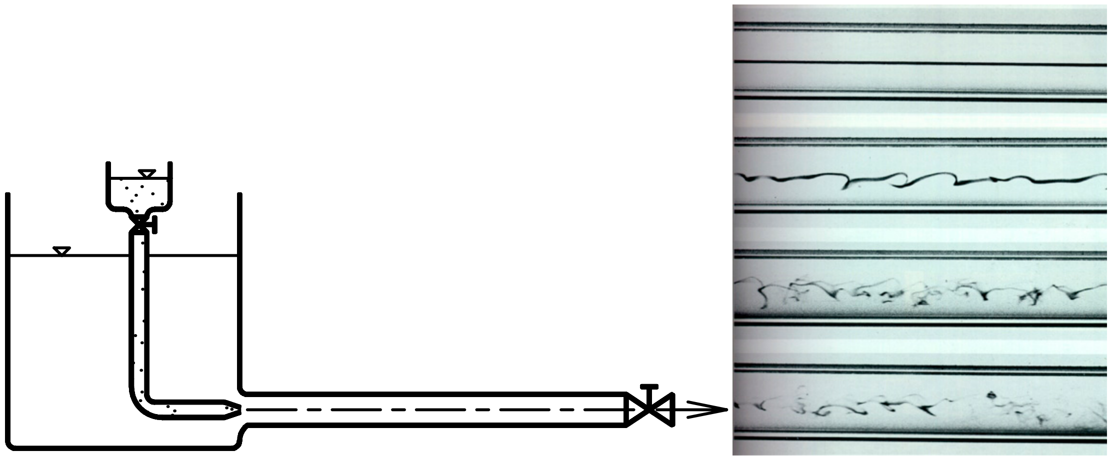
Friction loss over l: $\Delta p'[\mathrm{Pa}]\Rightarrow\frac{dp}{dz}=-\frac{\Delta p'}{l}$ $\Rightarrow$ $v_z=\frac{\Delta p'}{4\mu l}\left[R^2-r^2\right]\Rightarrow\tau=-\frac{\Delta p'}{2l}r$

$$v_{z\max}=\frac{\Delta p' R^2}{4\mu l}\;\Rightarrow\bar{v}=\frac{v_{z\max}}{2}\;\Rightarrow\bar{v}=\frac{1}{8}\frac{\Delta p'}{\mu l}R^2$$

$\boxed{\Delta p'=\frac{8\mu\bar{v}l}{R^2}}$. Wall shear stress: $\tau_{wall}=-\frac{\Delta p' R}{2l}$ $\quad 2R\pi\, l\,\tau_{wall}=R^2\pi\Delta p'$

25. Laminar and turbulent flows

Properties of turbulent flow



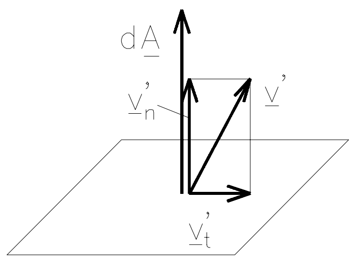
Transition depends on Reynolds number: $\mathrm{Re} = \frac{v d \rho}{\mu} \Rightarrow \mathrm{Re} \cong 2300$

time-average: , $\underline{\bar{v}} = \frac{1}{T}\int_0^T \underline{v}\,dt$ fluctuating velocity: $\underline{v}'$ $\left(\overline{\underline{v}'} = \frac{1}{T}\int_0^T \underline{v}'dt = 0\right)$

$$\underline{v} = \underline{\bar{v}} + \underline{v}'. \quad p = \bar{p} + p'.$$

Turbulence intensity: $Tu = \frac{\sqrt{\overline{(\underline{v}')^2}}}{|\underline{\bar{v}}|} = \frac{\sqrt{\overline{v_x'^2} + \overline{v_y'^2} + \overline{v_z'^2}}}{|\underline{\bar{v}}|}$

Apparent (Reynolds) stresses



$$\frac{\partial \bar{v}_x}{\partial t} + \frac{\partial(\bar{v}_x^2)}{\partial x} + \frac{\partial(\bar{v}_x \bar{v}_y)}{\partial y} + \frac{\partial(\bar{v}_x \bar{v}_z)}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\left(\frac{\partial^2 \bar{v}_x}{\partial x^2} + \frac{\partial^2 \bar{v}_x}{\partial y^2} + \frac{\partial^2 \bar{v}_x}{\partial z^2}\right) - $$

$$- \frac{\partial\left(\overline{v_x'^2}\right)}{\partial x} - \frac{\partial\left(\overline{v'_x v'_y}\right)}{\partial y} - \frac{\partial\left(\overline{v'_x v'_z}\right)}{\partial z}.$$

$$\int_A \underline{\bar{v}}\rho\underline{\bar{v}}d\underline{A} = -\int_A \bar{p}d\underline{A} + \int_V \rho\underline{g}dV - \int_A \overline{\underline{v}'\rho\underline{v}'}d\underline{A}$$

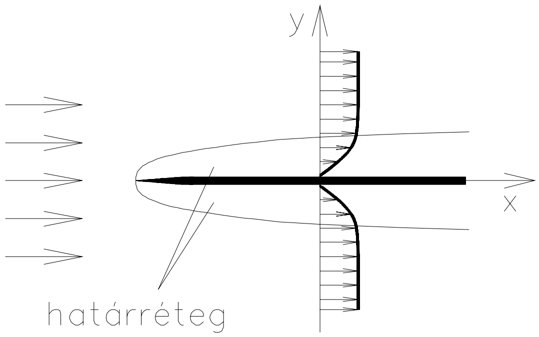
$$-\int_A \overline{\underline{v}'\rho\underline{v}'}d\underline{A} = -\int_A \overline{\underline{v}'_n \rho v'_n}dA - \int_A \overline{\underline{v}'_t \rho v'_n}dA .$$

$$\int_A \underline{\bar{v}}\rho\underline{\bar{v}}d\underline{A} = -\int_A \left[\bar{p} + \rho\left(\overline{v'^2_n}\right)\right]d\underline{A} + \int_V \rho\underline{g}dV - \int_A \rho\left(\overline{\underline{v}'_t v'_n}\right)dA$$

$p_\ell = \rho\left(\overline{v'^2_n}\right)$ apparent pressure rise $\tau_\ell = -\rho\left(\overline{v'_t v'_n}\right)$ apparent shear stress

$$-\frac{\partial\left(\overline{v_x'^2}\right)}{\partial x} - \frac{\partial\left(\overline{v'_x v'_y}\right)}{\partial y} - \frac{\partial\left(\overline{v'_x v'_z}\right)}{\partial z} = \frac{1}{\rho}\left(\frac{\partial \sigma_{xt}}{\partial x} + \frac{\partial \tau_{yxt}}{\partial y} + \frac{\partial \tau_{zxt}}{\partial z}\right)$$

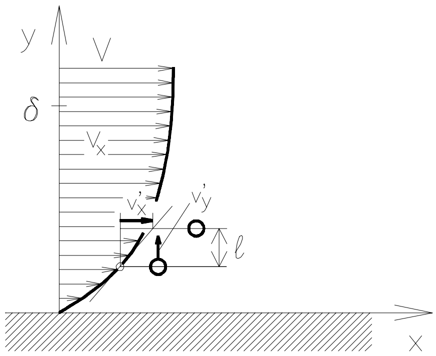
26. Boundary layers



$$v_z = 0, \frac{\partial(\ )}{\partial z} = 0, \ v_y << v_x \quad \frac{\partial(\ )}{\partial x} << \frac{\partial(\ )}{\partial y}$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = V \frac{dV}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$$



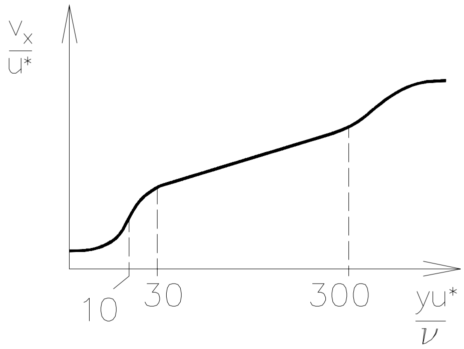
$$\tau = \tau_{yx\ell} = \rho \ell^2 \left| \frac{\partial v_x}{\partial y} \right| \frac{\partial v_x}{\partial y} = \mu_t \frac{\partial v_x}{\partial y}, \quad$$ 1 [m] mixing length

$$\mu_t = \rho \ell^2 \left| \frac{\partial v_x}{\partial y} \right|$$ eddy viscosity

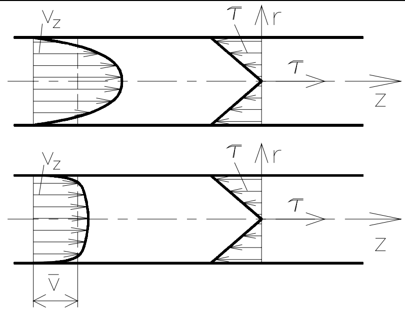
$$u^* = \sqrt{\frac{\tau_0}{\rho}}$$ [m/s] friction velocity

$$\boxed{\frac{v_x}{u^*} = \frac{1}{\kappa} \ln \frac{y u^*}{\nu} + K}$$ where $\kappa = 0.4$, $K = 5$

in viscous sublayer $\tau = \tau_0 = \mu \frac{\partial v_x}{\partial y}$, $\boxed{\frac{v_x}{u^*} = \frac{y u^*}{\nu}}$

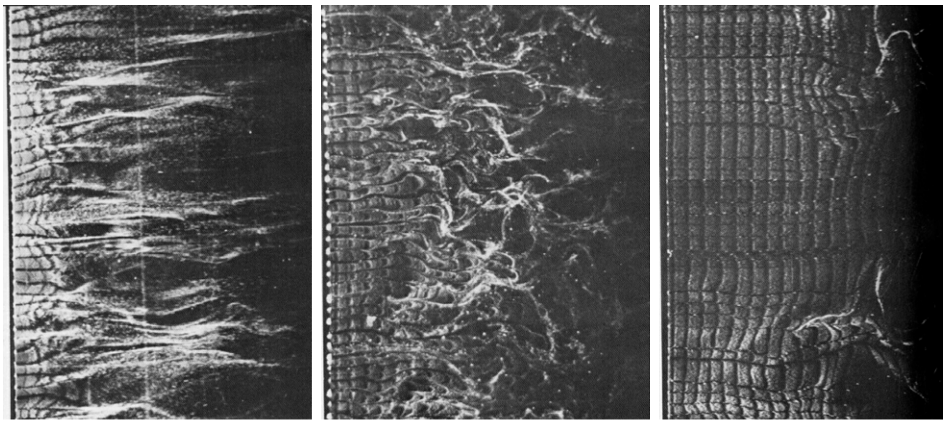


Velocity and shear stress distribution in laminar and turbulent pipe flows



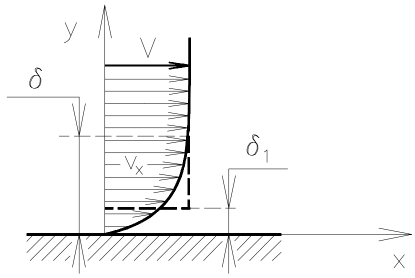
Characteristics of the flow in boundary layer

$$y^{+} = \frac{yu^{*}}{\nu}$$



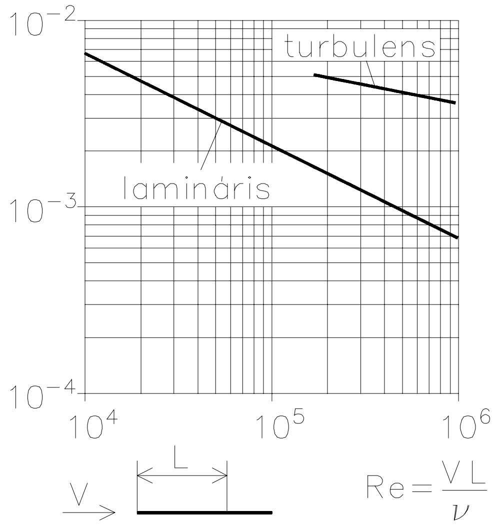
$y^{+}=2{,}7$ $y^{+}=101$ $y^{+}=407$

Displacement thickness

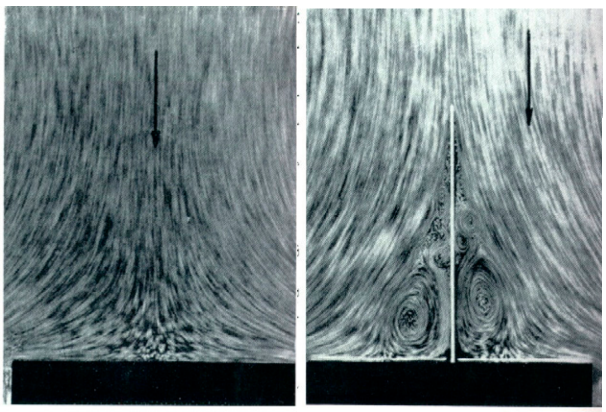


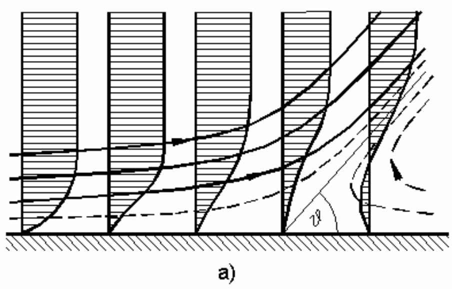
$$\delta_1 = \int_0^{\delta} \left(1 - \frac{v_x}{V} \right) dy \, .$$

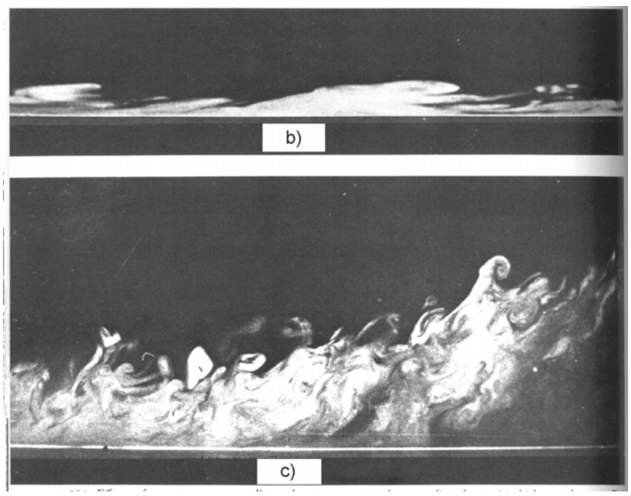
Friction coefficient $c'_f = \dfrac{\tau_0}{\frac{\rho}{2} V^2}$



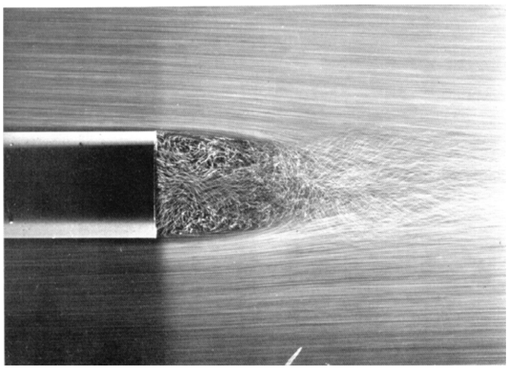
Boundary layer separation



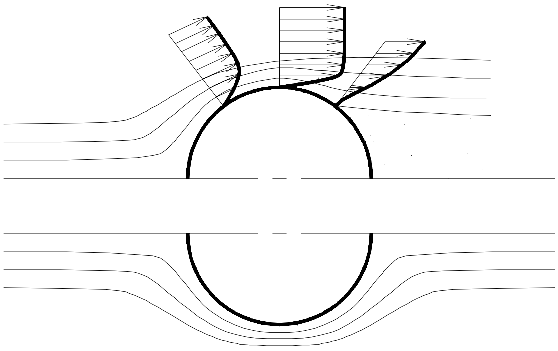


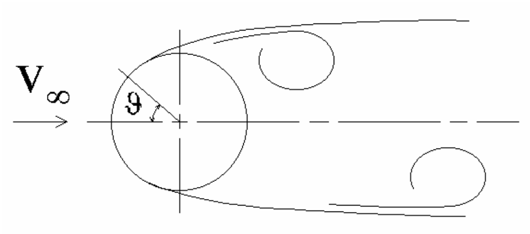


Separation bubble

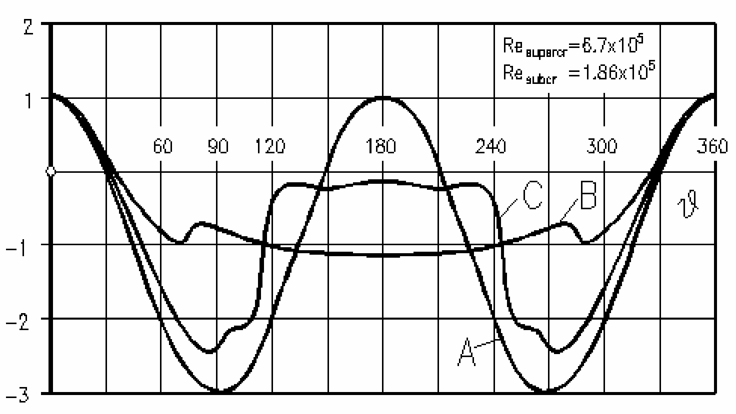


Flow past cylinders

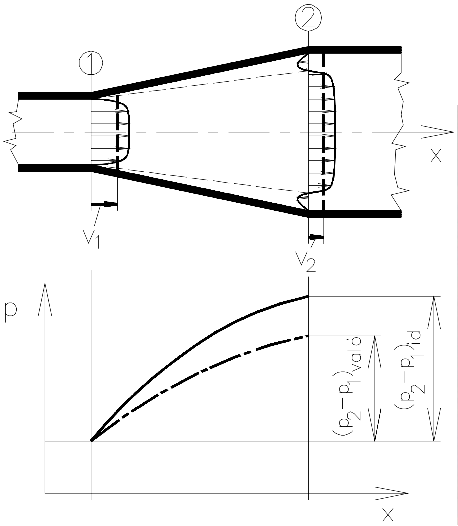


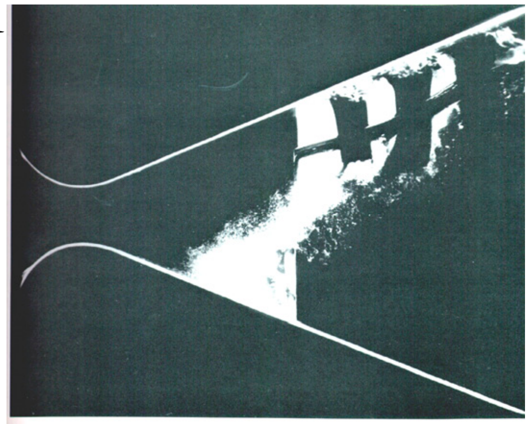


$$c_p = \frac{p - p_0}{\frac{\rho}{2} V^2}$$ pressure coefficient

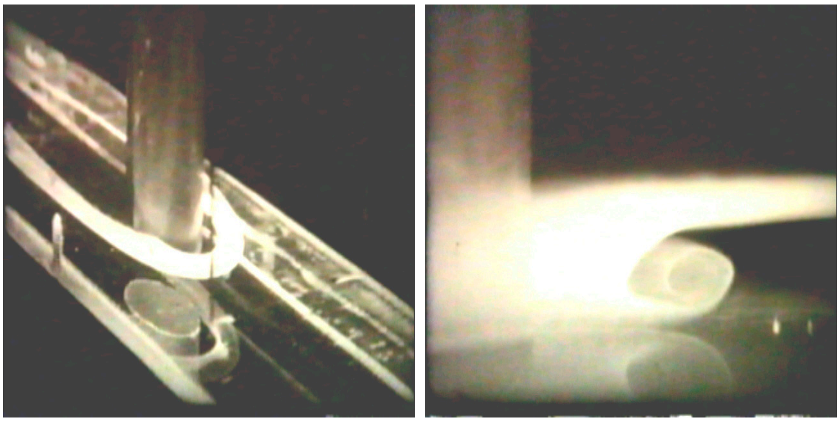


Flow in diffusers

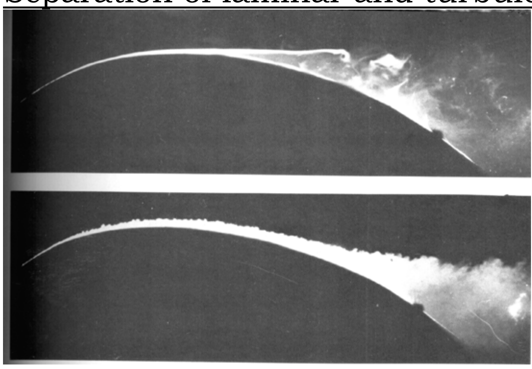




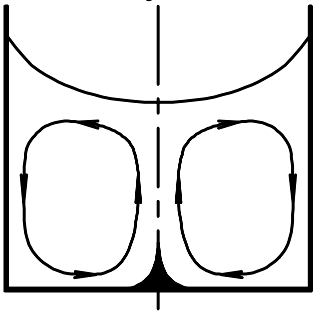
Horse-shoe vortex



Separation of laminar and turbulent boundary layers

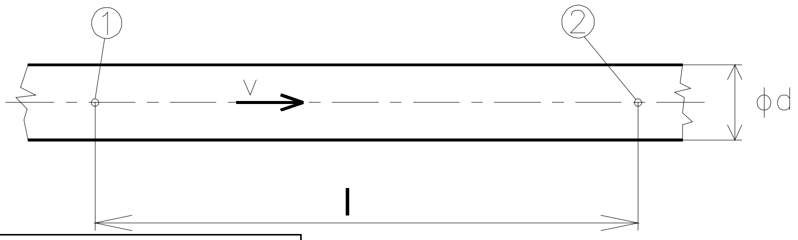


Secondary flow



27. Hidraulics

Bernoulli equation completed with friction losses



$$\rho\frac{v_1^2}{2}+p_1+\rho U_1=\rho\frac{v_2^2}{2}+p_2+\rho U_2+\Delta p'$$

Dimensional analyses

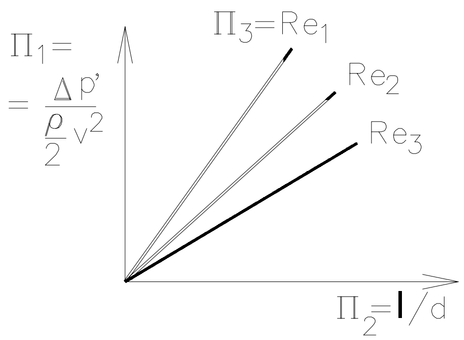
$$\Delta p'=f(\ell,\mu,\rho,d,v)$$

$[Q]=kg^{\alpha}m^{\beta}s^{\gamma}\Rightarrow Q_1,Q_2,......,Q_n.\Rightarrow F(Q_1,Q_2,......,Q_n)=0\Rightarrow\Pi_1,\Pi_2,\Pi_3,.....\Pi_{n-r}$, dimensionless parameters $\Rightarrow F(\Pi_1,\Pi_2,\Pi_3,.....\Pi_{n-r},)=0$:

$$\Pi=\Delta p'^{k_1}\ell^{k_2}\mu^{k_3}\rho^{k_4}d^{k_5}v^{k_6}.$$

$$\Pi_1=\frac{\Delta p'}{\frac{\rho}{2}v^2},\ \Pi_2=\ell/d,\ \Pi_3=Re=\frac{vd}{\nu}.$$

$$F(\Pi_1,\Pi_2,\Pi_3)=0$$



$$\Pi_1=f(\Pi_2,\Pi_3)$$

$$\frac{\Delta p'}{\frac{\rho}{2}v^2}=\lambda(Re)\frac{\ell}{d}\Rightarrow \boxed{\Delta p'=\frac{\rho}{2}v^2\frac{\ell}{d}\lambda(Re)}$$ where λ is the friction factor

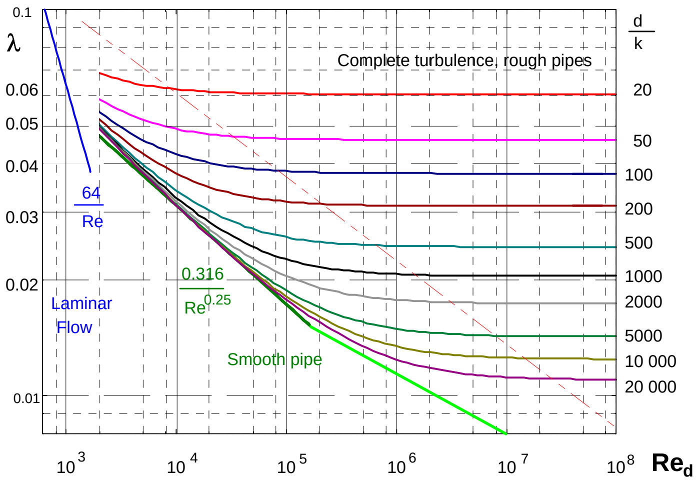
In case of laminar flow in pipe $\Delta p'=\frac{8\mu\ell}{R^2}v$, where $R=\frac{d}{2}\Rightarrow\Delta p'=\frac{\rho}{2}v^2\frac{\ell}{d}\frac{64\nu}{vd}$,

Since $\frac{vd}{\nu}=Re\Rightarrow\Delta p'=\frac{\rho}{2}v^2\frac{\ell}{d}\lambda_{lam}$ $\boxed{\lambda_{lam}=\frac{64}{Re}}$

Rough wall $k[m]$ diameter of sand particles, relative roughness $\Pi_4 = \frac{r}{k}$, where, r=d/2 pipe radius. Re > 2300 $Re \le Re_h$ $\frac{1}{\sqrt{\lambda_{turb}}} = 1.95\lg\left(Re\sqrt{\lambda_{turb}}\right) - 0.55$

$4000 \le Re \le 10^5$ Blasius formula $\boxed{\lambda_{turb} = \frac{0.316}{\sqrt[4]{Re}}}$

Moody diagram for determination of the friction factor λ for pipes



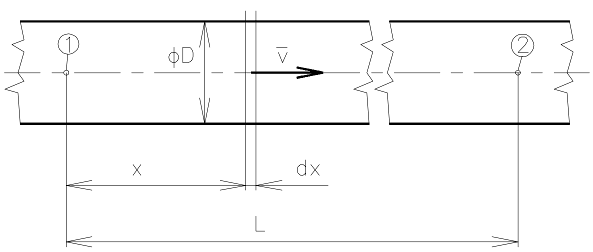
Wall shear stress

$$\Delta p' \frac{d^2\pi}{4} = \frac{\rho}{2} v^2 \frac{\ell}{d} \lambda \frac{d^2\pi}{4} = \tau_0 d\pi\ell \Rightarrow \tau_0 = \frac{\rho}{2} v^2 \frac{\lambda}{4}$$

Pipes of noncircular cross sections: $d_e = \frac{4A}{K}$, where $A\left[m^2\right]$ cross section, $K[m]$ wetted perimeter

$$\Delta p' = \frac{\rho}{2} v^2 \frac{\ell}{d_e} \lambda(Re) \text{ , where } Re = \frac{v d_e}{\nu}$$

Compressible flow in pipe

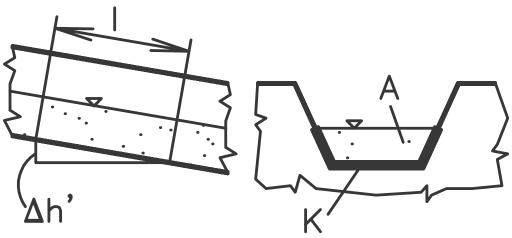


$$-dp = \frac{\rho}{2} v^2 \frac{dx}{D} \lambda, \bar{v} = \frac{q_m}{\rho A}, \rho = \frac{p}{RT} \Rightarrow -dp = \frac{q_m^2 RT}{2pA^2} \frac{\lambda}{D} dx \Rightarrow -\int_{p_1}^{p_2} p dp = \int_0^L \frac{q_m^2 RT\lambda}{2A^2 D} dx$$

Since $Re = \frac{\bar{v}D}{\nu} = \frac{q_m D}{\rho A \nu} = \frac{q_m D}{A\mu}$, $\mu = f(T)$ and $T \cong$ const. $\lambda \cong$ Const.

$$\frac{p_1^2 - p_2^2}{2} = \frac{q_m^2 RT\lambda L}{2A^2 D}\frac{\rho_1^2}{\rho_1^2} \Rightarrow \frac{p_1^2 - p_2^2}{2} = p_1 \frac{\rho_1}{2}\bar{v}_1^2 \frac{L}{D}\lambda \Rightarrow \boxed{\frac{p_1^2 - p_2^2}{2} = p_1 \Delta p'_{ink}}$$

Open channel flow



$\Delta h' = \frac{\bar{v}^2}{2g}\frac{\ell}{d_e}\lambda$, where $d_e = \frac{4A}{K}$, introducing $i = \frac{\Delta h'}{\ell}$ slope of the channel bottom

$\bar{v} = \sqrt{\frac{2gd_e}{\lambda}i} = C\sqrt{d_e i}$, where $C = \sqrt{\frac{2g}{\lambda}}$ Chézy equation, with $\lambda = 0.02 \sim 0.03$ Chézy coefficient $C \cong 28$.

Losses in pipe components

Losses of development of pipe flow

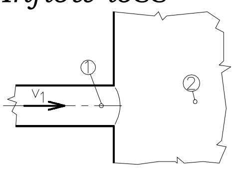
$$\Delta p'_{dev} = \frac{\rho}{2}v^2\zeta_{dev}$$

laminar flow: $\zeta_{dev,lam} \cong 1.2$, turbulent flow: $\zeta_{dev,torb} \cong 0.05$

Loss in Borda-Carnot extension

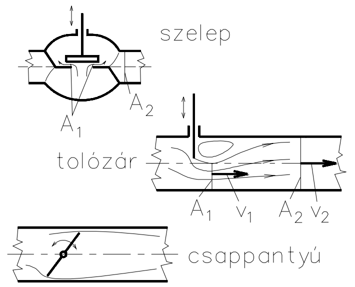
$$\Delta p'_{BC} = \frac{\rho}{2}(v_1 - v_2)^2$$

Inflow loss



$$\Delta p'_{in} = \frac{\rho}{2}(v_1 - 0)^2 = \frac{\rho}{2}v_1^2 .$$

Losses in valves

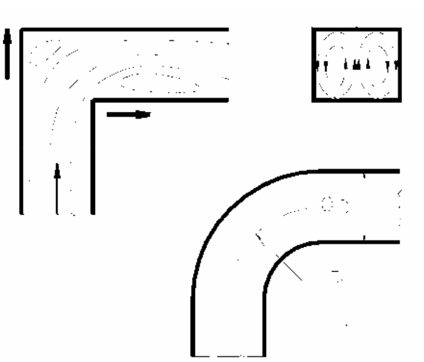


$$\Delta p'_v \cong \frac{\rho}{2} v_2^2 \left(\frac{v_1}{v_2} - 1 \right)^2, \zeta_v \cong \left(\frac{A_2}{A_1} - 1 \right)^2$$

Diffuser loss

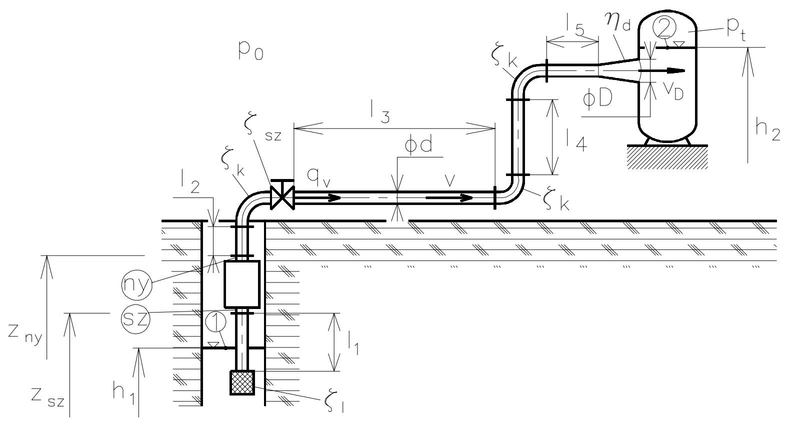
$\Delta p'_{diff} = (p_2 - p_1)_{id} - (p_2 - p_1)_{val} = (1 - \eta_d) \frac{\rho}{2} (v_1^2 - v_2^2)$ η_d diffuser efficiency

Losses in bends $\Delta p'_b = \frac{\rho}{2} v^2 \zeta_b$



<u>Applications</u>

Water supply system



Known parameters: d ℓ_1, ℓ_2 h_1, h_2 D $q_v \left[\frac{m^3}{s} \right]$, literature: $\zeta_l, \zeta_{sz}, \zeta_k$ η_d.

Calculation of pump head $H = \frac{\Delta p_{š}}{\rho g} + (z_{ny} - z_{sz})$ and performance $P_h = q_v \rho g H$

2 Bernoulli equations:

1) $\rho\frac{v_1^2}{2}+p_1+\rho U_1=\rho\frac{v_{sz}^2}{2}+p_{sz}+\rho U_{sz}+\Sigma\Delta p'_{sz}$, where $\sum\Delta p'_{sz}=\frac{\rho}{2}v^2\left(\zeta_1+\frac{\ell_1}{d}\lambda_1\right)$

2) $\rho\frac{v_{ny}^2}{2}+p_{ny}+\rho U_{ny}=\rho\frac{v_2^2}{2}+p_2+\rho U_2+\Sigma\Delta p'_{ny}$, where

$$\sum\Delta p'_{ny}=\frac{\rho}{2}v^2\left(\frac{\ell_2}{d}\lambda_2+\zeta_k+\zeta_{sz}+\frac{\ell_3}{d}\lambda_3+\zeta_k+\frac{\ell_4}{d}\lambda_4+\zeta_k+\frac{\ell_5}{d}\lambda_5\right)+$$

$$+(1-\eta_d)\frac{\rho}{2}\left(v^2-v_D^2\right)+\frac{\rho}{2}v_D^2.$$

Considering that $v_1=v_2=0$, $U_{ny}-U_{sz}=g(z_{ny}-z_{sz})$, $U_2-U_1=g(h_2-h_1)$, $p_1=p_0$, $p_2=p_t$ $\lambda_1=\lambda_2=....=\lambda$.

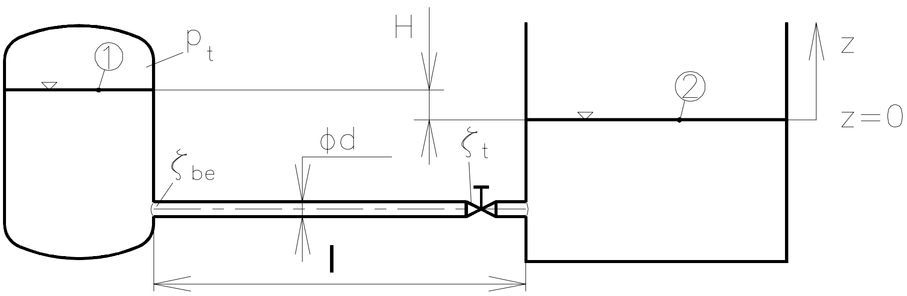
$$\Delta p_{š}=p_{nyš}-p_{szš}=p_t-p_0+\rho g(h_2-h_1)-\rho g(z_{ny}-z_{sz})+$$

$$+\frac{\rho}{2}v^2\left(\zeta_1+\zeta_{sz}+3\zeta_k+\frac{\Sigma\ell_i}{d}\lambda\right)+(1-\eta_d)\frac{\rho}{2}\left(v^2-v_D^2\right)+\frac{\rho}{2}v_D^2.$$

Continuity equation: $v d^2=v_D D^2$. $Re=\frac{vd}{\nu}\Rightarrow\lambda$

Flow in pipe connecting water tanks

Known parameters: d ℓ ζ_t. Calculation of $q_v\left[\frac{m^3}{s}\right]$



$$\rho\frac{v_1^2}{2}+p_1+\rho U_1=\rho\frac{v_2^2}{2}+p_2+\rho U_2+\Sigma\Delta p'$$

Considering $p_1=p_t$ and $p_2=p_0$, $U=gz$ és $z_2=0$, $z_1=H$, $v_1=v_2=0$ és a

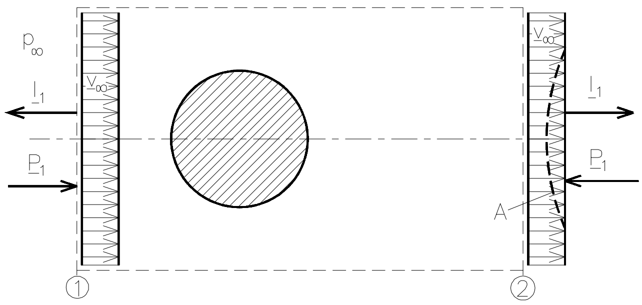
$$\Sigma\Delta p'_{sz}=\frac{\rho}{2}v^2\left(\zeta_{be}+\zeta_t+\frac{\ell}{d}\lambda+1\right)\Rightarrow p_t-p_0+\rho gH=\frac{\rho}{2}v^2\left(\zeta_{be}+\zeta_t+\frac{\ell}{d}\lambda+1\right)$$

$$v=\sqrt{\frac{2}{\rho}\frac{p_t-p_0+\rho gH}{(\zeta_{be}+\zeta_t+1)+\frac{\ell}{d}\lambda}}$$

$$v=\sqrt{\frac{A}{B+C\lambda}}$$

assuming $\lambda'\Rightarrow v'\Rightarrow Re'=\frac{v'd}{\nu}\Rightarrow\lambda''$–t .... $q_v=v\frac{d^2\pi}{4}$.

28. Aerodynamic forces and moments



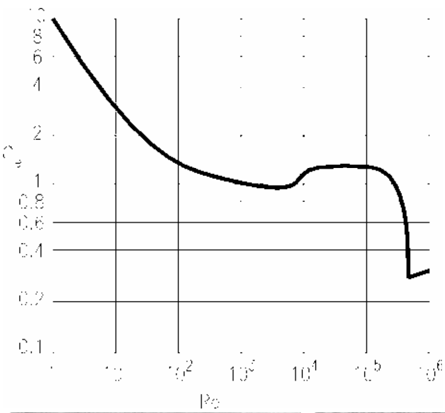
Inviscid fluid: $\left|\underline{p}_2\right| = \left|\underline{p}_1\right|$ Since $p_2 = p_1 = p_\infty$, $v_2 = v_1$, $\Rightarrow$ $\left|\underline{I}_2\right| = \left|\underline{I}_1\right|$. $-I_1 + I_2 = P_1 - P_2 - R_x$, $\Rightarrow$ $R_x = 0$

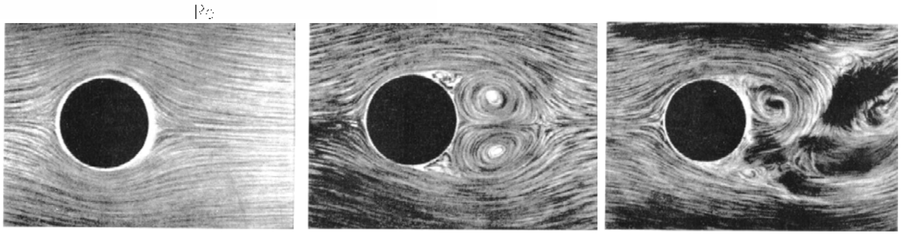
<u>Aerodynamic force acting on cylinder</u>

$f(F_d, v_\infty, \rho, \mu, d, \ell) = 0$ Dimensionless parameters: $\Pi_1 = c_d = \dfrac{F_d}{\frac{\rho}{2} v_\infty^2 \ell d}$ drag coefficient,

$\Pi_2 = Re = \dfrac{v_\infty d}{\nu}$ Reynolds number, $\Pi_3 = \dfrac{\ell}{d}$ relative length

$\Pi_3 = \dfrac{\ell}{d} = \infty$ 2D flows $\Rightarrow$ $\Pi_1 = f(\Pi_2)$, $c_d = f(Re)$

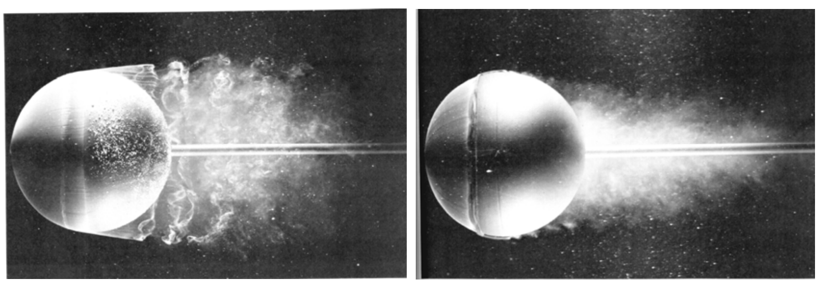




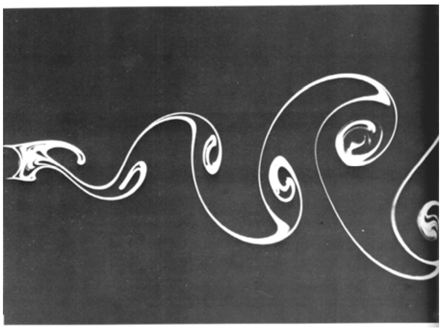
If Re is small, $F_d \sim \mu v_\infty$,

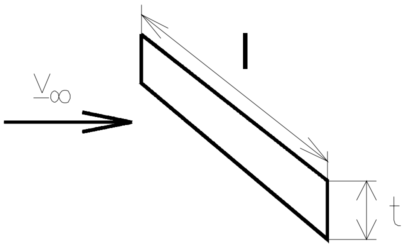
In case of larger Re $F_d \sim v_\infty^2$

Effect of laminar-turbulent BL transition.



Karman vortex street:



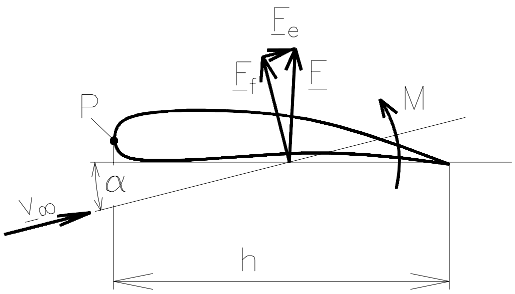


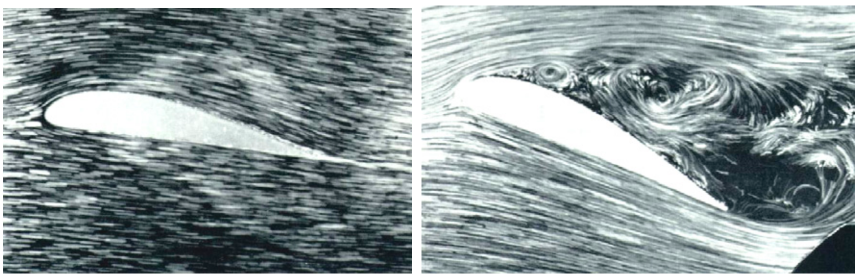
2D flow, $\frac{\ell}{t} \Rightarrow \infty$ $c_d = 2$. $F_d = (\overline{p}_f - \overline{p}_b)\ell t \Rightarrow c_d = \frac{\overline{p}_f - p_\infty}{\frac{\rho}{2} v_\infty^2} - \frac{\overline{p}_b - p_\infty}{\frac{\rho}{2} v_\infty^2} = \overline{c}_{pf} - \overline{c}_{pb}$

$c_{p\,max} = 1$ $\overline{c}_{pf} \cong 0.7 \Rightarrow \overline{c}_{pb} \cong -1.3$

3D effects: $\frac{\ell}{d} = \infty, 10, 1$ c_d = 2, 1.3, 1.1. In case of circular cylinder $c_d = 1.2$, 0.82 és 0.63 (Re=10^5).

Lift and drag on airfoils

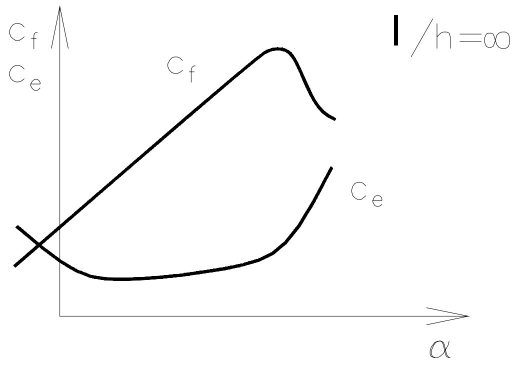




$|\underline{R}| = \rho v_\infty \Gamma \left[\frac{N}{m}\right]$ $c_l = \frac{F_l}{\frac{\rho}{2} v_\infty^2 A}$ lift $c_d = \frac{F_d}{\frac{\rho}{2} v_\infty^2 A}$ drag

$c_{Mp} = \frac{M_p}{\frac{\rho}{2} v_\infty^2 A h}$ pitching moment coefficient

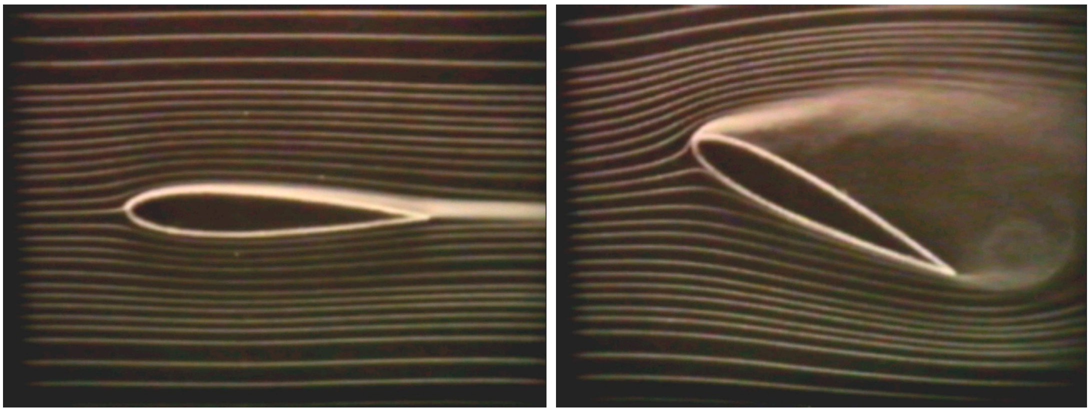
Lift and drag coefficients as function of angle of attack



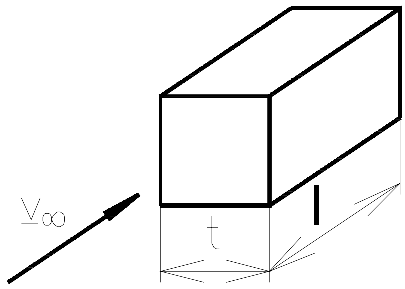
$c_e - c_D$, $c_f - c_L$

$c_{L\,max} \cong 1.2 \sim 1.8$, $dc_L/d\alpha = 2\pi$[rad],

Adverse pressure gradient can cause boundary layer separation



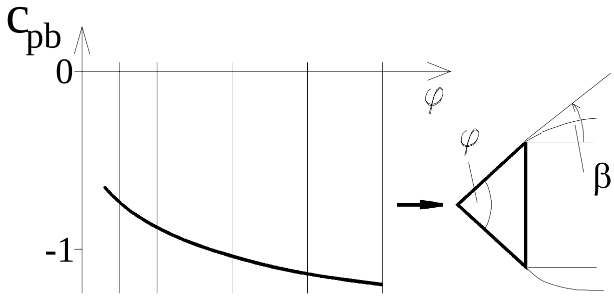
Drag acting on a prismatic bluff body



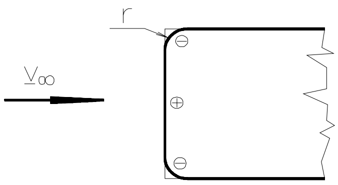
$c_D = \bar{c}_{pf} - \bar{c}_{pb} + 4\frac{\ell}{t}\bar{c}'_f$. Drag = forebody drag + base drag + side wall drag

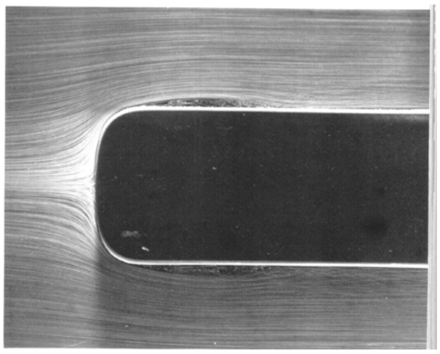
$\ell/t = 0$ and $\ell/t = 5$ c_e = 1.1 and 0.8.

Base pressure coefficient as function of angle between the undisturbed flow and the shear layer connecting to BL separation line.



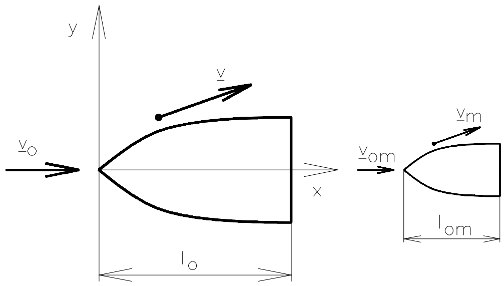
Reduction of forebody drag by rounding up of leading edges.





$\ell/t = 5 \quad c_D = 0.8 \Rightarrow 0.2$

29. Similarity of flows



Characteristic velocities, lengths, times of full scale prototype and model: v_0 and v_{0m}, l_0 and l_{0m}, $t_0 = \frac{\ell_0}{v_0}$ and $t_{0m} = \frac{\ell_{0m}}{v_{0m}}$.

$$\frac{\underline{v}}{v_0} = f\left(\frac{x}{\ell_0}, \frac{y}{\ell_0}, \frac{z}{\ell_0}, \frac{t}{t_0}\right) \text{ and } \frac{p}{\rho v_0^2} = F\left(\frac{x}{\ell_0}, \frac{y}{\ell_0}, \frac{z}{\ell_0}, \frac{t}{t_0}\right).$$

Conditions of similarity:

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \qquad \text{multiplied by } \frac{\ell_0}{v_0^2},$$

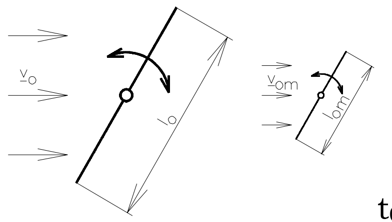
$$\frac{\partial\left(\frac{v_x}{v_0}\right)}{\partial\left(\frac{t}{\ell_0/v_0}\right)} + \frac{v_x}{v_0}\frac{\partial\left(\frac{v_x}{v_0}\right)}{\partial\left(\frac{x}{\ell_0}\right)} + .. = \frac{g_x\ell_0}{v_0^2} - \frac{\partial\left(\frac{p-p_0}{\rho v_0^2}\right)}{\partial\left(\frac{x}{\ell_0}\right)} + \frac{\nu}{v_0\ell_0}\left(\frac{\partial^2\left(\frac{v_x}{v_0}\right)}{\partial\left(\frac{x}{\ell_0}\right)^2} + ..\right) \qquad \text{dimensionless NS equation}$$

Two flows are similar if

a) they are described by the same dimensionless NS equation, i.e.

$\frac{g_x \ell_0}{v_0^2} = \frac{g_{xm} \ell_{0m}}{v_{0m}^2} \Rightarrow Fr = \frac{v_0}{\sqrt{g\ell_0}}$ Froude number, $\frac{\nu}{v_0 \ell_0} = \frac{\nu_m}{v_{0m} \ell_{0m}} \Rightarrow Re = \frac{v_0 \ell_0}{\nu}$ Reynolds number, $Re_m = Re \quad Fr_m = Fr$

b) the initial and boundary conditions are the same in dimensionless form (e.g. geometric similarity of prototype and model).



$t_0 = l_0/v_0$, $\frac{t_{pm}}{t_{0m}} = \frac{t_p}{t_0}$ i.e. $\frac{t_{pm} v_{0m}}{\ell_{0m}} = \frac{t_p v_0}{\ell_0}$.

$t_p = \frac{1}{f}$, f[1/s] frequency $Str = \frac{f\ell_0}{v_0}$ Strouhal number

<u>Dimensionless parameters as ratios of forces acting on unit mass</u>

inertial force: $F_T \sim \frac{v_0^2}{\ell_0}$

field of force: $F_G \sim g$

pressure force $F_p \sim \frac{(p - p_0)\ell_0^2}{\rho \ell_0^3} = \frac{(p - p_0)}{\rho \ell_0}$

viscous force: $F_S \sim \rho\nu \frac{v_0}{\ell_0} \frac{\ell_0^2}{\rho \ell_0^3} = \nu \frac{v_0}{\ell_0^2}$

surface tension force: $F_F \sim \frac{C}{\ell_0} \frac{\ell_0^2}{\rho \ell_0^3} = \frac{C}{\rho \ell_0^2}$

Reynolds number: $Re \sim \frac{F_T}{F_S} \sim \frac{v_0^2 / \ell_0}{\nu v_0 / \ell_0^2} = \frac{v_0 \ell_0}{\nu}$

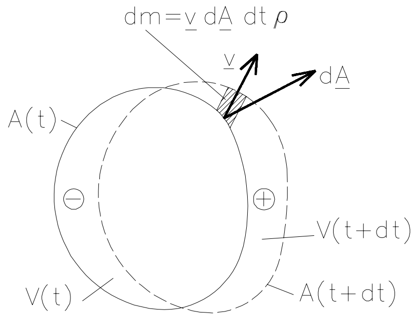
Froude number: $Fr \sim \sqrt{\frac{F_T}{F_G}} = \sqrt{\frac{v_0^2 / \ell_0}{g}} = \frac{v_0}{\sqrt{g\ell_0}}$

Euler number: $Eu \sim \frac{F_P}{F_T} \sim \frac{(p - p_0)/\rho/\ell_0}{v_0^2 / \ell_0} = \frac{p - p_0}{\rho v_0^2}$

Weber number: $We \sim \frac{F_F}{F_T} \sim \frac{C/\ell_0^2/\rho}{v_0^2/\ell_0} = \frac{C}{\rho v_0^2 \ell_0}$

30. Flow of compressible fluids

Energy equation



inviscid fluid, steady flow, no field of force, no heat transfer

$$\frac{d}{dt}\int_V \left(\frac{v^2}{2}+c_v T\right)\rho dV = -\int_A \underline{v} p d\underline{A} ,$$

where $c_v\left[\frac{J}{kgK}\right]$ constant-volume specific heat, $c_v T+\frac{p}{\rho}=h=c_p T$,

h [J/kg] az entalpia, c_p [J/kg/K] constant-pressure specific heat

$$\boxed{\frac{v^2}{2}+c_p T=\text{Const.}}$$ along streamline

Static, dynamic and total temperature

$$T+\frac{v^2}{2c_p}=T_t=\text{Const.}$$

where T (or T_{st}) [K] static temperature, $T_d=\frac{v^2}{2c_p}$[K] dynamic temperature, T_t[K] total temperature.

Energy equation ⇒ total temperature is constant along a streamline (steady flow of inviscid fluid)

Bernoulli equation for compressible gases

No viscous effects, and no heat transfer ⇒ isentropic flow:

$\frac{p}{\rho^\kappa}=\text{const.}=\frac{p_1}{\rho_1^\kappa}$ $,\kappa=\frac{c_p}{c_v}$ ratio of specific heats (isentropic exponent)

Bernoulli equation: $\frac{v_2^2-v_1^2}{2}=-\int_{p_1}^{p_2}\frac{dp}{\rho(p)}$ $\Rightarrow$ $v_2^2=v_1^2+\frac{2\kappa}{\kappa-1}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}\right].$

Velocity along a streamline: $\boxed{v_2=\sqrt{v_1^2+\frac{2\kappa}{\kappa-1}RT_1\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}\right]}}$ *

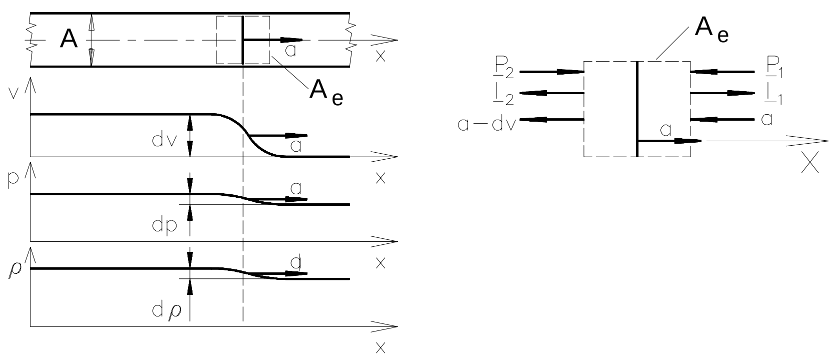
$\frac{p_1}{\rho_1^\kappa} = \text{const.} = \frac{p_2}{\rho_2^\kappa}$ and $\rho_1 = \frac{p_1}{RT_1}$ $\Rightarrow$ $\frac{p_2}{p_1} = \frac{\rho_2^\kappa}{\rho_1^\kappa} = \frac{p_2^\kappa T_1^\kappa R^\kappa}{R^\kappa T_2^\kappa p_1^\kappa} \Rightarrow \boxed{\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}}$

$\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\kappa}}}$, $\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\kappa-1}}}$. Inserting $\boxed{\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}}$ in * expression we obtain:

$v_2^2 = v_1^2 + \frac{2\kappa}{\kappa-1} RT_1\left[1 - \frac{T_2}{T_1}\right]$. $\frac{2\kappa R}{\kappa-1} = 2c_p, \kappa = \frac{c_p}{c_v}$ $\Rightarrow$ $v_2^2 = v_1^2 + 2c_p(T_1 - T_2)$ the energy equation $c_p T_1 + \frac{v_1^2}{2} = c_p T_2 + \frac{v_2^2}{2}$.

Discharge from a reservoir: $v = \sqrt{\frac{2\kappa}{\kappa-1} RT_t\left[1 - \left(\frac{p_e}{p_t}\right)^{\frac{\kappa-1}{\kappa}}\right]}$

Speed of sound



Integral momentum equation $\rho a^2 A - (\rho + d\rho)(a - dv)^2 A = (p + dp)A - pA$

$2a\rho dv - a^2 d\rho = dp$ Continuity: $(a - dv)(\rho + d\rho) = a\rho \Rightarrow \rho dv = a d\rho$.

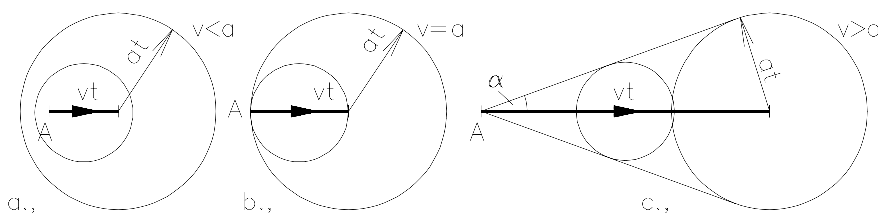
speed of sound: $\boxed{a = \sqrt{\frac{dp}{d\rho}}}$ In case of isentropic process $p = \frac{p_0}{\rho_0^\kappa}\rho^\kappa, \frac{dp}{d\rho} = \frac{p_0}{\rho_0^\kappa}\kappa\rho^{\kappa-1}$

speed of sound: $\sqrt{\frac{dp}{d\rho}} = \boxed{a = \sqrt{\kappa RT}}$

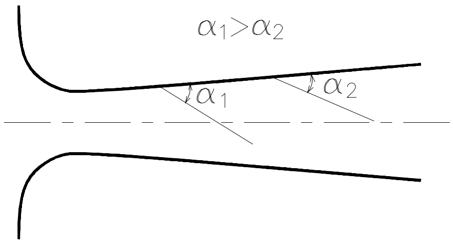
Additional conditions for similarity of flow of compressible fluids:

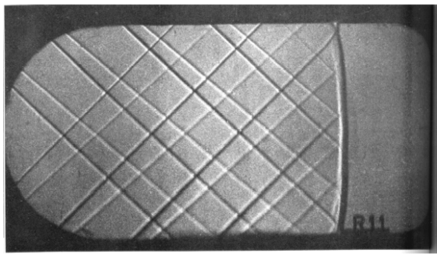
κ = identical and Mach number $Ma = \frac{v_0}{a_0}$ = identical

Propagation of pressure waves



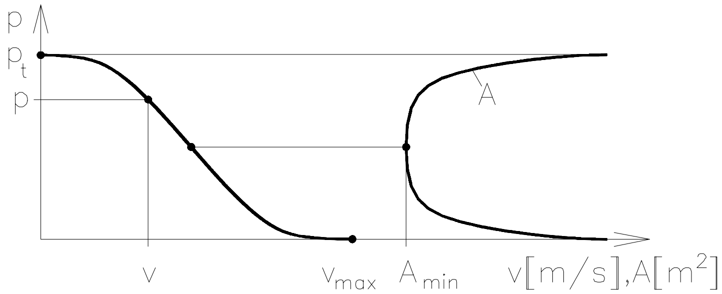
Mach cone $\sin\alpha = \frac{at}{vt} = \frac{a}{v} = \frac{1}{Ma}$





Release of a gas from a reservoir

$$v=\sqrt{\frac{2\kappa}{\kappa-1}RT_t\left[1-\left(\frac{p}{p_t}\right)^{\frac{\kappa-1}{\kappa}}\right]}.$$



$v_{max}=\sqrt{\frac{2\kappa}{\kappa-1}RT_t}$ tangent of the curve $v\frac{\partial v}{\partial e}=-\frac{1}{\rho}\frac{\partial p}{\partial e}+g_e \Rightarrow \frac{dp}{dv}=-\rho v$.

max $-\frac{dp}{dv} \Rightarrow \frac{d(\rho v)}{dv}=v\frac{d\rho}{dv}+\rho=0 \Rightarrow \rho\left[1-\frac{v^2}{dp/d\rho}\right]=\rho\left[1-\frac{v^2}{a^2}\right]=0$

in point of inflexion v = a, i.e. $Ma=1$

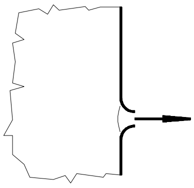
Since $q_m=\rho vA=-\frac{dp}{dv}A=\text{const.}$ at $q_m=\rho vA=-\frac{dp}{dv}A=\text{const.}$ (at $v=0$ and $p=0$) $A\to\infty$.

At $-\frac{dp}{dv}$ = max. the cross section A is the smallest (throat, critical area): Laval nozzle

$$T_t=T^*+\frac{a^{*2}}{2c_p}=T^*+\frac{\kappa RT^*}{2c_p}=T^*\left[1+\frac{c_p/c_v(c_p-c_v)}{2c_p}\right]=\frac{\kappa+1}{2}T^*$$

$$\frac{T^*}{T_t}=\frac{2}{\kappa+1}\ (=0.833),\ \frac{p^*}{p_t}=\left(\frac{T^*}{T_t}\right)^{\frac{\kappa}{\kappa-1}}=\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}}\ (=0.53)\ \frac{\rho^*}{\rho_t}=\left(\frac{T^*}{T_t}\right)^{\frac{1}{\kappa-1}}=\left(\frac{2}{\kappa+1}\right)^{\frac{1}{\kappa-1}}\ (=0.63).$$

Simple discharge nozzle



$p_e<p_t$ if $p_e/p_t>0.95$ $\rho\cong\text{const.}\Rightarrow v=\sqrt{\frac{2}{\rho}(p_t-p_e)}$

if $p_e/p_t < 0.95$ $\rho \neq$ const. $\Rightarrow$ $v = \sqrt{\dfrac{2\kappa}{\kappa-1} RT_t \left[1 - \left(\dfrac{p}{p_t}\right)^{\frac{\kappa-1}{\kappa}}\right]}$

if $p_e/p_t = 0.53$ (in case of $\kappa = 1.4$), $v^* = a^* = \sqrt{\kappa RT^*}$ where $T^* = 0.833 T_t$.
if $p_e/p_t < 0.53$ the process is same as in case of $p_e/p_t = 0.53$. ($p^* = 0.53 p_t$.)

Flow in Laval nozzle

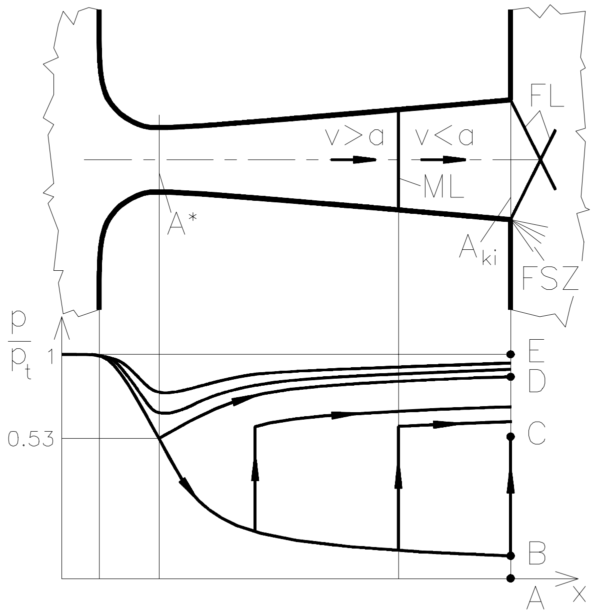
$$v\frac{\partial v}{\partial e} = -\frac{1}{\rho}\frac{\partial p}{\partial e} = -\frac{1}{\rho}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial e} = -\frac{1}{\rho}a^2\frac{\partial \rho}{\partial e}.$$

$$d\rho vA + \rho dvA + \rho v dA = 0 \Rightarrow \frac{d\rho}{\rho} + \frac{dv}{v} + \frac{dA}{A} = 0 \Rightarrow vdv = -a^2\frac{d\rho}{\rho}.$$

$$vdv = -a^2\frac{d\rho}{\rho} = a^2\left(\frac{dv}{v} + \frac{dA}{A}\right).$$

$$\frac{v^2}{a^2}\frac{dv}{v} = \frac{dv}{v} + \frac{dA}{A} \quad \Rightarrow \boxed{\left(Ma^2 - 1\right)\frac{dv}{v} = \frac{dA}{A}}$$

α/ if $Ma < 1$, in case of $dv/v > 0 \Rightarrow dA/A < 0$. At $dv/v < 0$) $dA/A > 0$
β/ If $Ma > 1$, in case of $dv/v > 0 \Rightarrow dA/A > 0$
γ/ if $dv/v > 0$, and $dA/A \Rightarrow Ma = 1$.
δ/ if $dA/A = 0$ and $Ma \neq 1$ the velocity is extreme.



continuity $q_m = \rho^* v^* A^* = \rho_{ki} v_{ki} A_{ki}$

$$0.63\rho_t\sqrt{\kappa R 0.83 T_t}\,A^* = A_{ki}\rho_t\left(\frac{p_e}{p_t}\right)^{\frac{1}{\kappa}}\sqrt{\frac{2\kappa}{\kappa-1}RT_t\left[1-\left(\frac{p_e}{p_t}\right)^{\frac{\kappa-1}{\kappa}}\right]}$$

Two isentropic solutions: points B and D.