6. Gas dynamics

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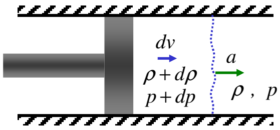
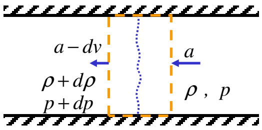
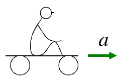
Speed of infinitesimal disturbances in still gas







Continuity:

$$A(a-dv)(\rho+d\rho)=a\,\rho\,A$$

$$a\,d\rho=\rho\,dv$$

Momentum theorem:

$$\sum \vec{I}=\sum \vec{P}$$

$$\underbrace{A\,\rho\,a}_{q_m}\underbrace{(a-(a-dv))}_{dv}=A\,dp$$

$$dp=\rho\,a\,dv$$

Allievi theorem

$$a^2=\frac{dp}{d\rho}$$

In steal	~5000 m/s
In water	~1500 m/s
In air	~340 m/s

Ideal gases

Equation of state: $\frac{p}{\rho}=RT$

We also assume that the specific heats are constant.

Internal energy: $u=c_v T$ Enthalpy: $h=u+\frac{p}{\rho}=c_p T$

Specific gas constant: $R=c_p-c_v=\frac{R_u}{M}$; $R_{air}=\frac{8314}{29}=287\left[\frac{\mathrm{J}}{\mathrm{kg\,K}}\right]$

Ratio of specific heats: $\gamma=\frac{c_p}{c_v}$ eg. for all diatomic gases:

$$\gamma=1.4$$

The speed of sound in ideal gases

We assume isentropic compression, which is very fast and the effect of the friction is negligible, thus:

$$\frac{p}{\rho^{\gamma}} = \text{const.}$$

$$\ln p - \gamma \ln \rho = \ln(\text{const.})$$

$$\frac{dp}{p} - \gamma \frac{d\rho}{\rho} = 0$$

$$\frac{dp}{d\rho} = \gamma \frac{p}{\rho} = \gamma R T$$

$$a = \sqrt{\gamma R T}$$

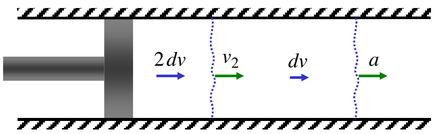
Eg. for air:

at 0 °C: a=331 m/s

at 20 °C: a=343 m/s

Nonlinear wave propagation

What if we generate another small disturbance?



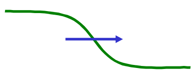
$v_2 > a$ because:

- The second wave propagates in a gas flow of dv velocity.
- The second wave propagates in a gas flow having a higher speed of sound: $p\uparrow \;\rightarrow\; T\uparrow \;\rightarrow\; a\uparrow$.

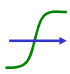
→ The second wave will catch up to the first wave.

Shock waves

A compression wave is steepening, and finally it becomes a **shock wave**:



Expansion waves behave in the opposite way:



- Treated as a discontinuity (finite jump) of the state variables (p, ρ, T and a).
- Propagates faster than the small disturbances. (Only shock waves can do so.)
- Deceleration of supersonic flows are generally caused by shock waves.
- It is a dissipative process. (Causes head losses.)

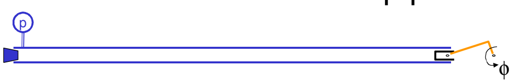
Analogy

Waves breaking in shallow water

Analogy

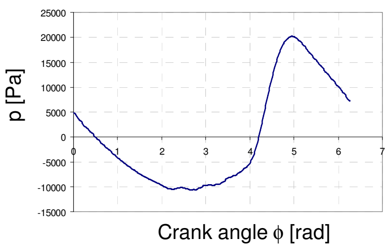
Hydraulic jump in a sink

Resonance in a closed pipe



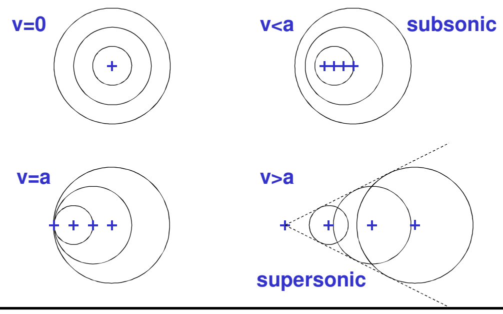
Pipe length:
6.05 m
Diameter:
36 mm
Piston displacement:
50 cm³.

At 29 Hz we measured:



Propagation of small disturbances in subsonic and in supersonic flow

Positions of an object having velocity v at time instants 0,-1,-2 and -3 seconds and also showing the wave fronts started in those instants:

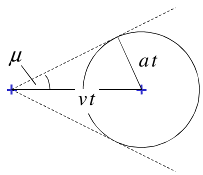


Application

Schlieren image of a gun fire

[http://www.phschool.com/science/science_news/articles/revealing_covert_actions.html]

Mach cone

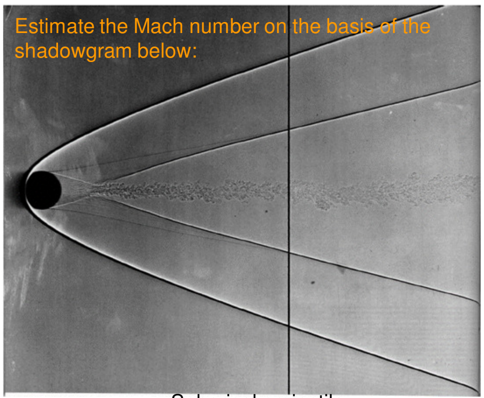


Mach number: $M = \frac{v}{a}$

Mach angle: $\mu = arc\,sin\left(\frac{a}{v}\right) = arc\,sin\left(\frac{1}{M}\right)$

Problem #6.1

Estimate the Mach number on the basis of the shadowgram below:

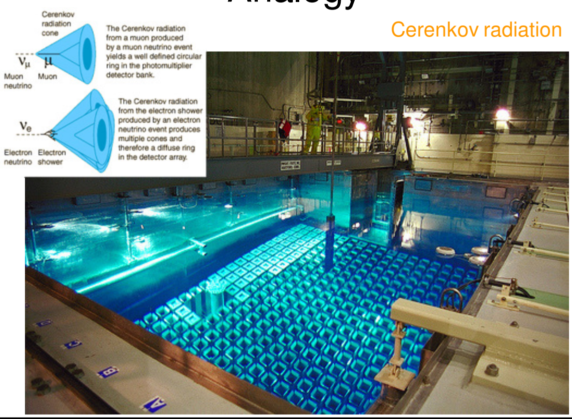


[An album of fluid motion] Spherical projectile

To the solution

Analogy

Cerenkov radiation

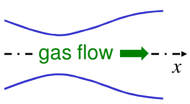


Variable cross-section channel (1)

Continuity: $$\frac{dA}{A}+\frac{dv}{v}+\frac{d\rho}{\rho}=0$$

Euler equation: $$v\,dv=-\frac{dp}{\rho}$$

Definition of a: $$a^2=\frac{dp}{d\rho}$$



A, v, p and ρ depend only on x

$$v^2\frac{dv}{v}=-\frac{dp}{\rho}\underbrace{\frac{d\rho}{dp}a^2}_{1}=-a^2\frac{d\rho}{\rho}$$

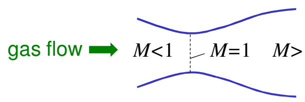
$$M^2\frac{dv}{v}=-\frac{d\rho}{\rho}=\frac{dA}{A}+\frac{dv}{v} \longrightarrow \left(M^2-1\right)\frac{dv}{v}=\frac{dA}{A}$$

Variable cross-section channel (2)

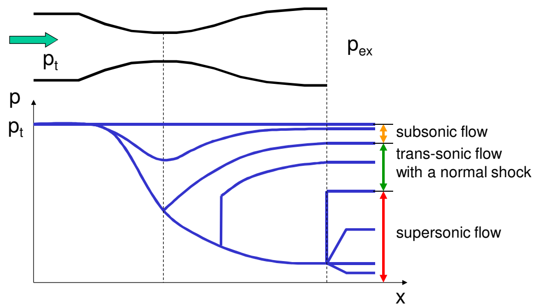
$$\left(M^2-1\right)\frac{dv}{v}=\frac{dA}{A}$$

	Acceleration	Deceleration
Subsonic $M<1$	Convergent	Divergent
Supersonic $M>1$	Divergent	Convergent

If $M=1$ then $dA=0$: the area has an extreme value (minimum).

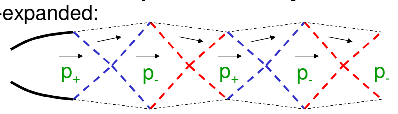


Laval nozzle

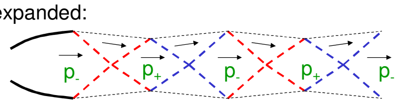


Supersonic jets

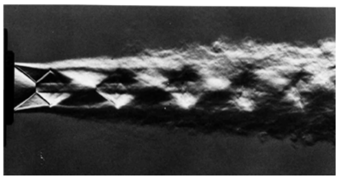
Under-expanded:



Over-expanded:



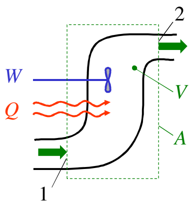
M=1.8



[An Album of Fluid Motion, 168]

Energy equation (1)

$$\frac{\partial}{\partial t}\int_V (u+\frac{v^2}{2})\rho\, dV + \oint_A (u+\frac{v^2}{2})\rho\,\vec{v}\, d\vec{A} = Q + W - \oint_A p\,\vec{v}\, d\vec{A}$$



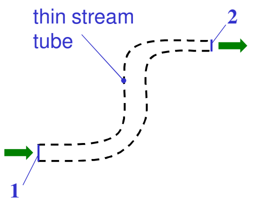
For steady state:

$$\oint_A (h+\frac{v^2}{2})\rho\,\vec{v}\, d\vec{A} = Q + W$$

Denoting the mass weighted average of the stagnation (total) enthalpy in cross-sections 1 and 2 by $h_{t,1}$ and $h_{t,2}$, it reads:

$$(h_{t,2}-h_{t,1})q_m = Q + W$$

Energy equation (2)



The stream tube can be regarded as a moving wall.

We apply the energy equation for steady flow under the following assumptions:
-the stream tube is thermally isolated (Q=0);
-the shear stress is 0 over the stream tube (W=0).

We obtain: $h_{t,2} = h_{t,1}$

Isentropic flow (1)

I. law of thermodynamics: $T\,ds = du + p\,d\left(\rho^{-1}\right)$

for an ideal gas: $T\,ds = c_v dT - \frac{p}{\rho^2}d\rho = c_v dT - RT\frac{d\rho}{\rho}$

for isentropic flow: $c_v\frac{dT}{T} = R\frac{d\rho}{\rho}$

$$\frac{R}{c_v} = \frac{c_p - c_v}{c_v} = \gamma - 1$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} \longleftarrow \frac{dT}{T} = (\gamma-1)\frac{d\rho}{\rho}$$

Isentropic flow (2)

$$\frac{dT}{T}=(\gamma-1)\frac{d\rho}{\rho}$$

$$\frac{dp}{p}=\frac{d\rho}{\rho}+\frac{dT}{T}$$

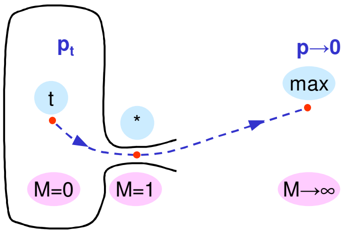
$$\frac{dT}{T}=(\gamma-1)\left[\frac{dp}{p}-\frac{dT}{T}\right]$$

$$\gamma\frac{dT}{T}=(\gamma-1)\frac{dp}{p}$$

$$\frac{T_2}{T_1}=\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Isentropic flow (3)

Reference states



Isentropic flow (5)

We can express temperature T as a function of M:

$$h_t=h+\frac{v^2}{2}$$

$$c_pT_t=c_pT+\frac{v^2}{2}$$

$$a^2=\gamma RT=\gamma c_p\left(1-\frac{1}{\gamma}\right)T=(\gamma-1)c_pT$$

$$\frac{a_t^2}{\gamma-1}=\frac{a^2}{\gamma-1}+\frac{v^2}{2}$$

$$\frac{a_t^2}{a^2}=\frac{T_t}{T}=1+\frac{\gamma-1}{2}M^2$$

Isentropic flow (6)

Local pressure and density can be expressed in terms of the Mach number through the isentropic relations:

$$\frac{p_t}{p} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_t}{\rho} = \left(\frac{T_t}{T}\right)^{\frac{1}{\gamma-1}} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

The critical ratios (for the state of M=1):

$$\frac{T_*}{T_t} = \frac{2}{\gamma+1} \qquad \frac{p_*}{p_t} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{\rho_*}{\rho_t} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

For γ=1.4: **0.83** **0.53** **0.63**

Isentropic flow (8)

Mass flow-rate: $q_m = \rho v A = \frac{\rho}{\rho_t}\rho_t M \frac{a}{a_t} a_t A$

$$q_m = M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\left(\frac{1}{\gamma-1}+\frac{1}{2}\right)}\rho_t a_t A$$

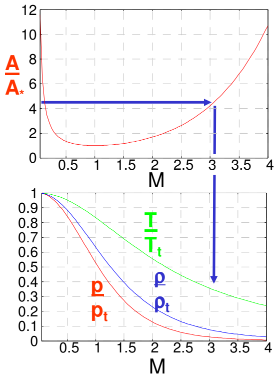
$$\frac{1}{\gamma-1}+\frac{1}{2} = \frac{2+\gamma-1}{2(\gamma-1)} = \frac{1}{2}\frac{\gamma+1}{\gamma-1}$$

$$q_m = M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{1}{2}\frac{\gamma+1}{\gamma-1}}\rho_t a_t A$$

||

$$q_m = \left(1+\frac{\gamma-1}{2}\right)^{-\frac{1}{2}\frac{\gamma+1}{\gamma-1}}\rho_t a_t A_* \longrightarrow \frac{A}{A_*} = f(M)$$

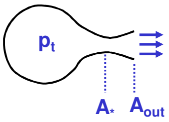
Isentropic flow (9)



$$\frac{A}{A_*} = \frac{M^{-1}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}}$$

The inverse of the above function also gives the Mach number for a given A/A* .

Problem #6.3



a) What is the optimum A_{out}/A_* ratio of the nozzle of a rocket thruster designed for near ground flight, if the chamber pressure p_t=10 bar_A, and γ=1.3. Please, use the gas tables!

b) Calculate the mass flow-rate for T_t=1300 K a, R=462 J/kg-K and A_{out}=20 cm²!

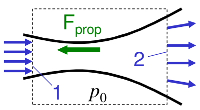
c) Please, calculate the thrust!

To the solution

Thrust function

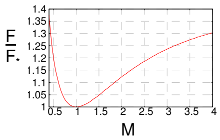
The momentum theorem for a variable cross-section steady channel flow reads:

$$F_{prop} = \left(p_2 + \rho_2 v_2^2\right)A_2 - \left(p_1 + \rho_1 v_1^2\right)A_1 + p_0\left(A_1 - A_2\right)$$



$$F = \left(p + \rho v^2\right)A$$

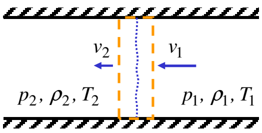
$$\frac{F}{F_*} = \frac{p + \rho v^2}{p_* + \rho_* v_*^2}\frac{A}{A_*} = \frac{p}{p_*}\frac{1+\gamma M^2}{1+\gamma}\frac{A}{A_*}$$



known functions of M. E.g:

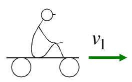
$$\frac{p}{p_*} = \frac{p_t}{p_*}\frac{p}{p_t} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} \Big/ \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

Normal shock waves (1)



4 unknowns.
We can eliminate one by using:

$$\frac{p_2}{\rho_2} = RT_2$$



A steady flow is observed!

Continuity:
$$v_1 \rho_1 A = v_2 \rho_2 A$$

Momentum low:
$$\left(p_1 + \rho_1 v_1^2\right)A = \left(p_2 + \rho_2 v_2^2\right)A$$

Energy equation:
$$\left(c_p T_1 + \frac{v_1^2}{2}\right)\rho_1 v_1 A = \left(c_p T_2 + \frac{v_2^2}{2}\right)\rho_2 v_2 A$$

Normal shock waves (2)

Mach number was the key to isentropic flows ...

... we should try to solve this problem for $M_2(M_1)$.

$$\rho_1 v_1 = \ldots \longrightarrow \frac{p_1}{RT_1} M_1 (\gamma R T_1)^{1/2} = \ldots$$

$$p_1 + \rho_1 v_1^2 = \ldots \longrightarrow p_1\left(1 + \frac{\rho_1 v_1^2}{p_1}\right) = \ldots \longrightarrow p_1\left(1 + \gamma \frac{v_1^2}{a_1^2}\right) = \ldots$$

$$p_1\left(1 + \gamma M_1^2\right) = \ldots$$

$$c_p T_1 + \frac{v_1^2}{2} = \ldots \longrightarrow T_1\left(1 + \frac{\gamma R v_1^2}{2 c_p a_1^2}\right) = \ldots \longrightarrow T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right) = \ldots$$

Normal shock waves (3)

(a) $\frac{p_1}{RT_1} M_1 (\gamma R T_1)^{1/2} = \ldots$

(b) $p_1\left(1 + \gamma M_1^2\right) = \ldots$

(c) $T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right) = \ldots$

$a*b^{-1}*c^{0.5}$

$$\frac{M_1}{1+\gamma M_1^2}\sqrt{1 + \frac{\gamma-1}{2} M_1^2} = \frac{M_2}{1+\gamma M_2^2}\sqrt{1 + \frac{\gamma-1}{2} M_2^2}$$

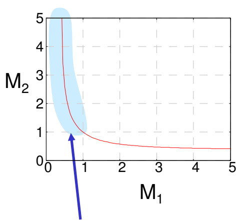
$$M_1^2\left(1 + \frac{\gamma-1}{2} M_1^2\right)\left(1+\gamma M_2^2\right)^2 = M_2^2\left(1 + \frac{\gamma-1}{2} M_2^2\right)\left(1+\gamma M_1^2\right)^2$$

It is a quadratic formula for M_2^2

We can arrange it into the polynomial form:

$$M_2^4(\ldots) + M_2^2(\ldots) + (\ldots) = 0$$

Normal shock waves (4)



$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma}{\gamma - 1} M_1^2 - 1}$$

This branch belongs to an expansion shock.
Is it valid?

Normal shock waves (5)

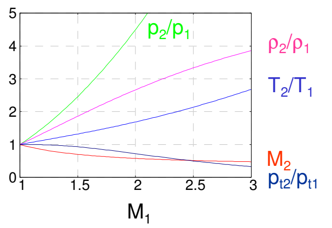
Pressure ratio: (b) ⟶ $\frac{p_2}{p_1}=\frac{1+\gamma M_1^2}{1+\gamma M_2^2}=f(M_1)$

Temperature ratio: (c) ⟶ $\frac{T_2}{T_1}=\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}=g(M_1)$

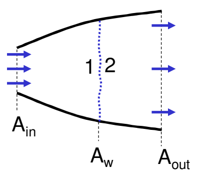
$$\frac{\rho_2}{\rho_1}=\frac{p_2}{p_1}\left(\frac{T_2}{T_1}\right)^{-1}=h(M_1)$$

Normal shock waves (6)

$$\frac{p_{t2}}{p_{t1}}=\frac{\frac{p_{t2}}{p_2}\frac{p_2}{p_1}}{\frac{p_{t1}}{p_1}}=\frac{\left(\frac{T_{t2}}{T_2}\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{T_{t1}}{T_1}\right)^{\frac{\gamma}{\gamma-1}}}\frac{p_2}{p_1}=\left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}}\frac{p_2}{p_1}$$



Problem #6.4



There is a strong stationary normal shock in a divergent channel at the cross-section characterized by A_w.

$\gamma=1.4$ $\quad M_{in}=2$

$p_{in}=100\,kPa_A$ $\quad T_{in}=270\,K$

$A_w/A_{in}=2$ $\quad A_{out}/A_{in}=3$

a) Calculate the Mach number at the outlet (M_{out})!

b) Please, determine the outlet pressure (p_{out})!

To the solution