1. Vorticity transport

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Acceleration of a fluid parcel

Motion of clouds: rudder, satelite

Velocity components: $\vec{v}(t,\vec{r}) = u(t,x,y,z)\vec{i} + v(t,x,y,z)\vec{j} + w(t,x,y,z)\vec{k}$

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

For a fluid parcel: $\frac{dx}{dt} = u,\quad \frac{dy}{dt} = v,\quad \frac{dz}{dt} = w$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \vec{v}\cdot\nabla u$$
$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \vec{v}\cdot\nabla v$$
$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \vec{v}\cdot\nabla w$$

$$\rightarrow \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}$$

- $\nabla\vec{v}$: velocity gradient tensor
- $\frac{\partial \vec{v}}{\partial t}$: local acceleration
- $\vec{v}\cdot\nabla\vec{v}$: convective acceleration

Navier-Stokes equation

$$\rho = const, \quad \nu = const$$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g} + \nu\,\Delta\vec{v} \qquad \left[\frac{N}{kg}\right]$$

- $-\frac{1}{\rho}\nabla p$: pressure force
- $\vec{g}$: body-force
- $\nu\,\Delta\vec{v}$: shear

Continuity

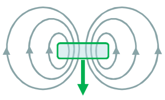
$$\frac{\partial}{\partial t}\int_V \rho\, dV + \oint_A \rho \underline{v} \cdot dA = 0$$ which yields: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

accumulated mass — **mass outflux**

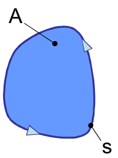
If ρ=const: $\nabla \cdot \underline{v} = 0$ thus $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Stream tubes can form closed tubes or have both ends on boundaries.

Eg. absolute streamlines around a moving object:



Vortices



Circulation:

$$\Gamma = \oint_s \vec{v} \cdot d\vec{s}$$

Vorticity:

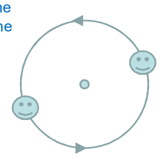
$$\vec{\omega} = \nabla \times \vec{v} = \begin{pmatrix} w'_y - v'_z \\ u'_z - w'_x \\ v'_x - u'_y \end{pmatrix}$$

From the Stokes theorem we also know that: $\Gamma = \int_A \vec{\omega} \cdot d\vec{A}$

Can we have circulation without vorticity?

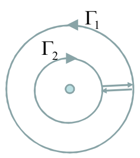
Free vortex

Vorticity is 2 times the angular velocity of the fluid parcel.



$$\omega_\perp = 0$$

Vorticity is concentrated in the center.



$$\Gamma_1 = -\Gamma_2$$

$$2r\pi\, v = \text{const.}$$

$$v = \frac{\text{const.}}{r}$$

Thomson theorem

If the the integration path S is a fluid line of a perfect fluid, then

$$\frac{d\Gamma}{dt}=0$$

Circulation is proportional wih the lift force.

How vorticity is produced?

Evolution of vorticity

Vorticity transport equation: $\nabla\times(\text{Navier}-\text{Stokes})$

Let's derive it in 2D! ω is a scalar in 2D: $\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$

By taking the curl of N-S equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right)+\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)+\frac{\partial g_y}{\partial x}+\nu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)+\frac{\partial g_x}{\partial y}+\nu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}+{*}{*}{*}=0+0+\nu\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)$$

$${*}{*}{*}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Because of the continuity: 0= 0=

Curl of the convective acceleration term in 3D flow

$$\nabla\times(\vec{v}\cdot\nabla\vec{v})=\nabla\times\left[\begin{pmatrix}u'_x & u'_y & u'_z\\ v'_x & v'_y & v'_z\\ w'_x & w'_y & w'_z\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}\right]=$$

$$\vec{\omega}=\nabla\times\begin{pmatrix}u\\ v\\ w\end{pmatrix}=\begin{pmatrix}w'_y-v'_z\\ u'_z-w'_x\\ v'_x-u'_y\end{pmatrix}$$

$$=\begin{pmatrix}w''_{xy}-v''_{xz} & w''_{yy}-v''_{yz} & w''_{zy}-v''_{zz}\\ u''_{xz}-w''_{xx} & u''_{yz}-w''_{yx} & u''_{zz}-w''_{zx}\\ v''_{xx}-u''_{xy} & v''_{yx}-u''_{yy} & v''_{zx}-u''_{zy}\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}+$$

$$+\begin{pmatrix}w'_xu'_y-v'_xu'_z+w'_yv'_y-v'_yv'_z+w'_zw'_y-v'_zw'_z\\ u'_xu'_z-w'_xu'_x+u'_yv'_z-w'_yv'_x+u'_zw'_z-w'_zw'_x\\ v'_xu'_x-u'_xu'_y+v'_yv'_x-u'_yv'_y+v'_zw'_x-u'_zw'_y\end{pmatrix}$$

$$\vec{\omega}=\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{pmatrix}w'_y-v'_z\\ u'_z-w'_x\\ v'_x-u'_y\end{pmatrix}$$

$$\nabla\times\left(\vec{v}\cdot\nabla\vec{v}\right)=$$

$$\begin{pmatrix}\alpha'_x & \alpha'_y & \alpha'_z\\ \beta'_x & \beta'_y & \beta'_z\\ \gamma'_x & \gamma'_y & \gamma'_z\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}+\begin{pmatrix}w'_xu'_y-v'_xu'_z-u'_xw'_y+u'_xv'_z\\ u'_yv'_z-w'_yv'_x-v'_yu'_z+v'_yw'_x\\ v'_zw'_x-u'_zw'_y-w'_zv'_x+w'_zu'_y\end{pmatrix}+\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\left(u'_x+v'_y+w'_z\right)$$

$$\vec{\omega}\cdot\nabla\vec{v}=\begin{pmatrix}u'_xw'_y-u'_xv'_z+u'_yu'_z-u'_yw'_x+u'_zv'_x-u'_zu'_y\\ v'_xw'_y-v'_xv'_z+v'_yu'_z-v'_yw'_x+v'_zv'_x-v'_zu'_y\\ w'_xw'_y-w'_xv'_z+w'_yu'_z-w'_yw'_x+w'_zv'_x-w'_zu'_y\end{pmatrix}$$

$$\nabla\times\left(\vec{v}\cdot\nabla\vec{v}\right)=\vec{v}\cdot\nabla\vec{\omega}-\vec{\omega}\cdot\nabla\vec{v}+\vec{\omega}\nabla\cdot\vec{v}$$

Vorticity transport

(ρ and ν are constants)

$$\frac{d\vec{v}}{dt}=\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}=-\frac{1}{\rho}\nabla p+\ \vec{g}\ +\nu\Delta\vec{v} \qquad \nabla\times\ldots$$

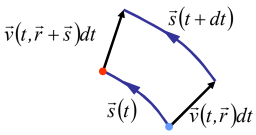
$$\frac{d\vec{\omega}}{dt}=\frac{\partial\vec{\omega}}{\partial t}+\vec{v}\cdot\nabla\vec{\omega}=\ 0\ \ +\nabla\times\vec{g}+\nu\Delta\vec{\omega}-\vec{\omega}\nabla\cdot\vec{v}+\vec{\omega}\cdot\nabla\vec{v}$$

vortex transport; 0, if g is irrotational; vortex diffusion; 0; vortex stretching

$$\frac{d\underline{\omega}}{dt}=\nu\Delta\underline{\omega}+\underline{\omega}\cdot\nabla\underline{v}$$

What is vortex stretching?

Evolution of a fluid line of elementary length



$$\vec{v}(t,\vec{r}+\vec{s})-\vec{v}(t,\vec{r})=\vec{s}\cdot\nabla\vec{v}$$

$$\frac{d\vec{s}}{dt}=\vec{s}\cdot\nabla\vec{v}$$

Vorticity transport equation for irrotational body force and zero viscosity:

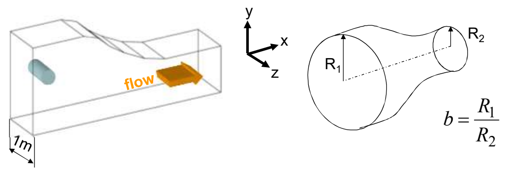
$$\frac{d\vec{\omega}}{dt}=\vec{\omega}\cdot\nabla\vec{v}$$

The direction of $\underline{s}$ is arbitrarily chosen.

Both vectors evolve according to the same transport equation, hence, in inviscid flow, the vorticity vector behaves in the same way as an infinitesimal fluid line element. (Helmholtz)

Thus, $\underline{\omega}$ will grow, when the fluid line is stretched.

Problem #1.1



Compare a 2D confuser (of slab symmetry) with an axial symmetric confuser:

- What components of the vorticity vector are non-zero?
  Use cylindrical coordinates (x,r,ϕ) in the axisymmetric case!

- In what proportion does the length of a fluid element change?

- In what proportion will the components of the vorticity change if the vortex diffusion is negligable?

To the solution

Vortex diffusion

The vorticity transport equation for a 2D flow of a constant property Newtonian fluid:

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \vec{v} \cdot \nabla \omega = \nu \Delta \omega \qquad \nu = \frac{\mu}{\rho} \left[\frac{m^2}{s} \right]$$

kinematical viscosity

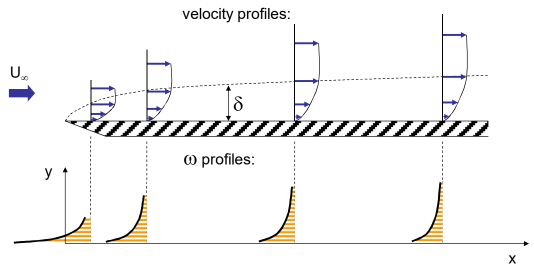
Is in full analogy with the heat transport equation:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = a \Delta T \qquad a = \frac{\lambda}{\rho c} \left[\frac{m^2}{s} \right]$$

heat diffusion coefficient

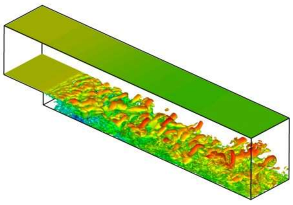
The kinematical viscosity can be regarded as a vorticity diffusion coefficient.
These two phenomena are in full analogy.

Boundary layer over a flat plate



Vorticity is continuously produced on the wall surface due to the no-slip condition, and it is conducted into the main stream by the viscosity.

The role of advection



http://www.computationalfluiddynamics.com.au/cfd-turbulence-part5-scale-resolving-simulations_srs/

Summary

The vorticity transport equation for incompressible fluids reads:

$$\frac{d\vec{\omega}}{dt} = \nabla \times \vec{g} + \nu \, \Delta\vec{\omega} + \vec{\omega} \cdot \nabla\vec{v}$$

Origin of vorticity:
- Boundary conditions (wall shear)
- Non conservative forces (eg. Coriolis force)

Redistribution of vorticity:
- Advection
- Vortex stretching
- Vortex diffusion