Compressible Flows

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1D isentropic flows

Unsteady isentropic flow in a constant cross-section pipe.
Eg. in an exhaust pipe.

Continuity:
$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0$$

Euler equation:
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Isentropic relation:
$$\frac{p}{\rho^{\gamma}}=\frac{p_0}{{\rho_0}^{\gamma}}$$

p_0 and ρ_0 are the pressure and density in the reference state.
p, ρ, u are unknown functions of x and t.

Introduction of the sound speed "a" as a new field variable

Only one state variable can be chosen in isentropic system.
We can use the speed of sound "a" to express the pressure (p) and density (ρ).
Both "u" and "a" do have the dimension of m/s.

$$\frac{p}{\rho^{\gamma}}=\frac{p_0}{{\rho_0}^{\gamma}}$$

$$ln(p)-\gamma\, ln(\rho)=0$$

$$\frac{dp}{p}=\gamma\frac{d\rho}{\rho}$$

$$a^2=\left.\frac{\partial p}{\partial\rho}\right|_{s=áll.}=\gamma\frac{p}{\rho}=\gamma\rho^{\gamma-1}\frac{p}{\rho^{\gamma}}=\gamma\frac{p_0}{{\rho_0}^{\gamma}}\rho^{\gamma-1}$$

$$2\,ln(a)=(\gamma-1)ln(\rho)+ln\left(\gamma\frac{p_0}{{\rho_0}^{\gamma}}\right)$$

$$2\frac{da}{a}=(\gamma-1)\frac{d\rho}{\rho}$$

$$\frac{\partial a}{\partial p}=\frac{\gamma-1}{2\gamma}\frac{a}{p}$$

$$\frac{\partial a}{\partial\rho}=\frac{\gamma-1}{2}\frac{a}{\rho}$$

We reformulate the governing equations

Continuity:
$$\frac{\partial\rho}{\partial t}\frac{\partial a}{\partial\rho}+u\frac{\partial\rho}{\partial x}\frac{\partial a}{\partial\rho}+\rho\frac{\partial u}{\partial x}\frac{\gamma-1}{2}\frac{a}{\rho}=0$$

$$\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial x}+\frac{\gamma-1}{2}a\frac{\partial u}{\partial x}=0 \qquad (1)$$

Euler equation:
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}\underbrace{\frac{\partial a}{\partial p}\frac{2\gamma}{\gamma-1}\frac{p}{a}}_{1}=0$$

a²

$$\frac{\gamma-1}{2}\frac{\partial u}{\partial t}+\frac{\gamma-1}{2}u\frac{\partial u}{\partial x}+a\frac{\partial a}{\partial x}=0 \qquad (2)$$

$$\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial x}+\frac{\gamma-1}{2}a\frac{\partial u}{\partial x}=0 \qquad (1)$$

$$\frac{\gamma-1}{2}\frac{\partial u}{\partial t}+\frac{\gamma-1}{2}u\frac{\partial u}{\partial x}+a\frac{\partial a}{\partial x}=0 \qquad (2)$$

(1) + (2)
$$\frac{\partial}{\partial t}\underbrace{\left(a+\frac{\gamma-1}{2}u\right)}_{\alpha}+(u+a)\frac{\partial}{\partial x}\underbrace{\left(a+\frac{\gamma-1}{2}u\right)}_{\alpha}=0$$

$$\frac{\partial\alpha}{\partial t}+(u+a)\frac{\partial\alpha}{\partial x}=0$$

α=const. in the direction of C_+=dx/dt=u+a.

(1) - (2)
$$\frac{\partial}{\partial t}\underbrace{\left(a-\frac{\gamma-1}{2}u\right)}_{\beta}+(u-a)\frac{\partial}{\partial x}\underbrace{\left(a-\frac{\gamma-1}{2}u\right)}_{\beta}=0$$

$$\frac{\partial\beta}{\partial t}+(u-a)\frac{\partial\beta}{\partial x}=0$$

β=const. in the direction of C_-=dx/dt=u-a

Characteristics

C_+ and C_- are the characteristic directions. α and β are Rieman invariants.

u and a can be expressed in terms of α and β.

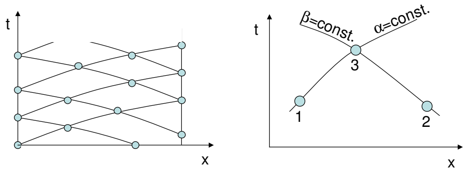
$$\alpha=a+\frac{\gamma-1}{2}u \qquad \beta=a-\frac{\gamma-1}{2}u$$

$$a=\frac{\alpha+\beta}{2} \qquad u=\frac{\alpha-\beta}{\gamma-1}$$

Every field variable can than be expressed in terms of a:

$$\left(\frac{a}{a_0}\right)^2=\frac{T}{T_0}=\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}=\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

Numerical solution



$\alpha_3 = \alpha_1$

$\beta_3 = \beta_2$ → $a_3 = \dfrac{\alpha_3+\beta_3}{2}$ $\quad u_3 = \dfrac{\alpha_3-\beta_3}{\gamma-1}$

$$x_3 - x_1 = 0.5[(u_3+a_3)+(u_1+a_1)](t_3-t_1)+o(\Delta t^2)$$

$$x_3 - x_2 = 0.5[(u_3-a_3)+(u_2-a_2)](t_3-t_2)+o(\Delta t^2)$$

t_3, x_3 can be calculated.

Finite volume method

The density based approach.

Continuity: $\dfrac{\partial \rho}{\partial t}+\dfrac{\partial \rho u}{\partial x}=0$

Eq.of motion: $\dfrac{\partial \rho u}{\partial t}+\dfrac{\partial(\rho u^2+p)}{\partial x}=0$

Equation of state: $p=\rho R T$

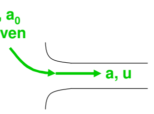
Energy eq.: $\dfrac{\partial \rho e}{\partial t}+\dfrac{\partial(\rho u e+p u)}{\partial x}=0$ $\qquad e=c_v T+\dfrac{u^2}{2}$

In vector format: $\dfrac{\partial \underline{U}}{\partial t}+\dfrac{\partial \underline{F}}{\partial x}=\underline{Q}$

$$\underline{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho e\end{bmatrix} \qquad \underline{F}=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho u e+p u\end{bmatrix} \qquad \underline{Q}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Boundary conditions

Inflow:



the energy equation

$$T_0 = T+\frac{u^2}{2c_p}=\frac{a^2}{\gamma R}+\frac{u^2}{2c_p}$$

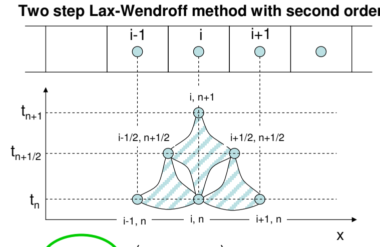
$$T_0=\frac{1}{\gamma R}\left(\frac{\alpha+\beta}{2}\right)^2+\frac{1}{2c_p}\left(\frac{\alpha-\beta}{\gamma-1}\right)^2$$

Either α or β is already given. (Along the outrunning characteristic curve.)
The other quantity can be expressed from the above equation.

Outflow: $a_0 = a = \dfrac{\alpha+\beta}{2}$

Closed pipe: $u=0$ → $\dfrac{\alpha-\beta}{\gamma-1}=0$ → $\alpha=\beta$

Two step Lax-Wendroff method with second order accuracy:



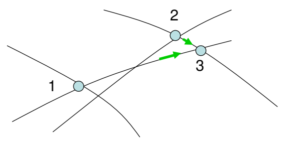
Step 1: $$\frac{U_{i+1/2}^{n+1/2}-\left(U_i^n+U_{i+1}^n\right)/2}{\Delta t/2}+\frac{F_{i+1}^n-F_i^n}{\Delta x}=\frac{Q_i^n+Q_{i+1}^n}{2}$$

When U is known ρ, u and e can be calculated. Eg. ρ=(ρu)/u
p is than obtained from the equation of state.
F and Q values can than be calculated at the time level n+1/2.

Step 2: $$\frac{U_i^{n+1}-U_i^n}{\Delta t}+\frac{F_{i+1/2}^{n+1/2}-F_{i-1/2}^{n+1/2}}{\Delta x}=\frac{Q_{i-1/2}^{n+1/2}+Q_{i+1/2}^{n+1/2}}{2}$$

The problems...

- The numerical resolution depend on the actual physical properties, therefore it can become very coarse in some regions.
- The characteristic curves running in the same direction can intersect each other.



This is an explicit time marching scheme. Only conditionally stable.
According to the linear stability theory:

$$\Delta t=\sigma\frac{\Delta x}{a+|u|} \qquad \sigma\le 1 \quad \text{Courant number}$$

Strong oscillations can take place in the presence of shockwaves.
Fluxes must be corrected by using some upwinding or artificial viscosity.

A similar approach in FLUENT: density based solver + explicit formulation (time integration). The multi step time integration method implemented in FLUENT allows somewhat larger Courant number. (The default value is σ=1.)

Specification of the boundary conditions:
the method of characteristics can be used at the domain boundaries.
(There are other approaches too.)