The solution of pressure velocity coupling

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Governing equations for incompressible flows

An iterative method will be used for solving the discrete forms of the governing equations.
Segregated iteration. (Separate equation is solved for every field variable, in which every other field variable is treated as a constant value.)

Continuity: $$\nabla\cdot\vec{v}=0$$

Navier-Stokes equation:

$$\frac{\partial u}{\partial t}+\nabla\cdot(u\vec{v})=-\frac{\partial p/\rho_0}{\partial x}+\nabla\cdot(\mathbf{v}\nabla u)+g_x$$

$$\frac{\partial v}{\partial t}+\nabla\cdot(v\vec{v})=-\frac{\partial p/\rho_0}{\partial y}+\nabla\cdot(\mathbf{v}\nabla v)+g_y$$

$$\frac{\partial w}{\partial t}+\nabla\cdot(w\vec{v})=-\frac{\partial p/\rho_0}{\partial z}+\nabla\cdot(\mathbf{v}\nabla w)+g_z$$

This system is not suitable for segragated iteration. For example:

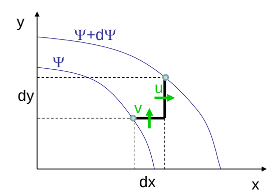
$$u_P^{n+1}\leftarrow\quad\frac{u_P^{n+1}-u_P^n}{\Delta t}+A_P u_{i,P}^n+\sum_{\ell\ szomszédra}A_\ell u_{i,\ell}^n=-\frac{1}{\rho_0}\nabla\Big|_x p^n+g_x$$??

How can we calculate p^n and how will the continuity be fulfilled for u^{n+1}?

Two possible solutions

- Ψ-ω method
  Eliminates the pressure from the equation of motion by the introduction of a potential function.
- Pressure correction method
  A new equation for the pressure field is solved instead of the continuity eq.

Stream function (Ψ)



Volume flow-rate in 1 m wide domain.

No flow through Ψ=const. lines, therefore iso-lines of Ψ are streamlines.

Now, we look at two close points on different streamlines:

$$d\psi=u\,dy-v\,dx$$

The total differential of Ψ:

$$d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy$$

Definition of Ψ (for 2D flows): $$\frac{\partial\psi}{\partial x}=-v\quad és\quad\frac{\partial\psi}{\partial y}=u$$

Any Ψ fulfills the continuity: $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial y\partial x}=0$$

Ψ in 3D

Ψ is a vector field in 3D cases: **Vector potential** $$\underline{v}=\nabla\times\underline{\psi}$$

It is conform with the original definition:

$$\underline{v}=\begin{pmatrix}\frac{\partial\psi_z}{\partial y}-\frac{\partial\psi_y}{\partial z}\\ \frac{\partial\psi_x}{\partial z}-\frac{\partial\psi_z}{\partial x}\\ \frac{\partial\psi_y}{\partial x}-\frac{\partial\psi_x}{\partial y}\end{pmatrix}\rightarrow$$

therefore, for 2D flows: $$u=\frac{\partial\psi}{\partial y}\quad és\quad v=-\frac{\partial\psi}{\partial x}$$

Continuity is automatically fulfilled, also in 3D: $$\nabla\cdot\underline{v}=\nabla\cdot\nabla\times\underline{\psi}\equiv 0$$

In 2D we reduce the number of unknown field variables: (u,v →Ψ).
This advantage will be lost in 3D.

Vorticity (ω)

Vorticity in 3D: $$\underline{\omega}=\nabla\times\underline{v}$$

$$\underline{\omega}=\begin{pmatrix}\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\\ \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\end{pmatrix}$$

In 2D ω is a scalar field (having only the z component)

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

ω can be expressed in terms of Ψ: $$\omega=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}$$

this is a **Poisson's equation for the Ψ**.

An important special case: $$\underline{\omega}=0$$

Only the Laplace equation need to be solved for Ψ: $$\Delta\psi=0$$

The vorticity transport equation in 2D

We take the curl of the Navier-Stokes equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)+\#\#\#=-\frac{\partial^2 p/\rho_0}{\partial x\partial y}+\mathbf{v}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}\right)\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)+\#\#\#=-\frac{\partial^2 p/\rho_0}{\partial x\partial y}+\mathbf{v}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)\right)$$

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}+\#\#\#=0+\mathbf{v}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=(\nabla\cdot\underline{v})\omega=0$$

We obtain a usual transport equation with convection and conduction terms:

$$\frac{d\omega}{dt}=\mathbf{v}\,\Delta\omega$$

The conduction coefficient is the kinematic viscosity.

The Ψ-ω method for steady flow

The Poisson equation for Ψ in a 2D case :

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega$$

The vorticity transport equation:

$$\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}=\mathbf{v}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)$$

Segregated iteration:

$$\psi^0,\omega^0\xrightarrow{Poisson}\psi^1,\omega^0\xrightarrow{ÖTE}\psi^1,\omega^1\xrightarrow{Poisson}\psi^2,\omega^1\ \ldots$$

Boundari cond. for Ψ:
- Inlet: BC of 1st kind.
- Outlet: BC of second kind (Neumann BC).

for ω:
- Inlets and walls: BC of 1st kind.
- Outlet: BC of second kind (Neumann BC).

Problem: we cannot impose pressure boundary conditions.
(Pressure field is unknown.)

Pressure based solution
The pressure equation

Continuity: $\nabla\cdot\underline{u}=0$

The equation of motion (g=0 is assumed): $\frac{\partial\underline{u}}{\partial t}+\nabla\cdot(\underline{u}\otimes\underline{u})=-\nabla(p/\rho_0)+\mathbf{v}\Delta\underline{u}$

Shorthand notations: $P=p/\rho_0$ and $\underline{f}$.

$$\frac{\partial\underline{u}}{\partial t}=\underline{f}-\nabla P$$

Let's take the divergence of the equation of motion by assuming: $\frac{\partial}{\partial t}\nabla\cdot\underline{u}=0$

$$\Delta P=\nabla\cdot\underline{f}$$

This is a Poisson equation for the pressure.

This is useful for calculating the pressure field in Ψ-ω methods.

Can the Poisson equation for the pressure be a substitute for the continuity equation?

Let's apply the Euter method for the numerical integration of the equation of motion:

$$u_i^{n+1}=u_i^n+\Delta t\left(f_i^n-\tilde{\nabla}\big|_i P^n\right)$$

and we apply the numerical Laplacian for discretizing the Poisson equation for the pressure:

$$\tilde{\Delta}P^n=\tilde{\nabla}\cdot f_i^n$$

With given right hand side and given boundary conditions, this is a linear system of algebraic equations. Let 's suppose, we solve this for P^n, then we update the velocity field. Now, let's check the divergence of the updated velocity field:

$$\tilde{\nabla}\cdot u_i^{n+1}=\tilde{\nabla}\cdot u_i^n+\Delta t\left(\tilde{\nabla}\cdot f_i^n-\tilde{\Delta}P^n\right)$$

≈0 only approximate solution is possible!

The numerical error of the solution of the Poisson equation will be accumulated in the continuity equation.

The accumulation of numerical errors can be avoided

Not the Poisson equation is important, but the continuity.

Instead of the original Poisson equation we solve the following:

$$\tilde{\Delta}P^n=\tilde{\nabla}\cdot f_i^n+\frac{1}{\Delta t}\tilde{\nabla}\cdot u_i^n$$

then we update the velocity by using the eq. of motion:

$$u_i^{n+1}=u_i^n+\Delta t\left(f_i^n-\tilde{\nabla}\big|_i P^n\right)$$

Now we check the divergence of the new velocity field:

$$\tilde{\nabla}\cdot u_i^{n+1}=\Delta t\left[\frac{1}{\Delta t}\tilde{\nabla}\cdot u_i^n+\tilde{\nabla}\cdot f_i^n-\tilde{\Delta}P^n\right]$$

≈0 this is what we solve for P^n.

The error of continuity equation is limited by the error of solution of the discrete Poisson equation in the last time step.

Projection method

The same method with different notations:

1-st step we evaluate: $u^*_i=u_i^n+\Delta t\ f_i^n$ — u*: pseudo-velocity

2-nd step **we solve**: $\tilde{\Delta}P^n=\frac{1}{\Delta t}\tilde{\nabla}\cdot u^*_i$ → P^n

3-rd step we evaluate: $u_i^{n+1}=u^*_i-\Delta t\tilde{\nabla}\big|_i P^n$

Let's check it! $\tilde{\nabla}\cdot u_i^{n+1}=\Delta t\left[\frac{1}{\Delta t}\tilde{\nabla}\cdot u^*_i-\tilde{\Delta}P^n\right]$

this (=0) is solved in the 2-nd step!

Steady flows

Explicit discretization in time does not fulfill an important practical requirement:
The method is only conditionally stabile, therefore time step size is limited. If steady state is slowly achieved, we need to make a high number of time steps.

P-u iteration for steady flow (1)

We want to fulfill in step n+1 the following with the highest possible accuracy:

$$A_P u_{i,P}^{n+1} + \sum A_\ell u_{i,\ell}^{n+1} = Q_i - \tilde{\nabla}\Big|_i P^{n+1} \qquad \text{és} \qquad \tilde{\nabla} \cdot u_i^{n+1} = 0$$

Only the old pressure value can be used... (the continuity is not accurately fulfilled in this stage)
u^n is used as an initial value for u^*.

$$A_P u_{i,P}^{*} + \sum A_\ell u_{i,\ell}^{*} = Q_i - \tilde{\nabla}\Big|_i P^{n} \rightarrow u_i^*$$

1-st step

$$u_{i,P}^{*} = \frac{Q_i - \sum A_\ell u_{i,\ell}^{*}}{A_P} - \frac{1}{A_P}\tilde{\nabla}\Big|_i P^{n}$$

$$u_{i,P}^{*} = \breve{u}_{i,P} - \frac{1}{A_P}\tilde{\nabla}\Big|_i P^{n}$$

u^{n+1} is calculated from an approximate formula from the new pressure field:

$$u_{i,P}^{n+1} \approx \breve{u}_{i,P} - \frac{1}{A_P}\tilde{\nabla}\Big|_i P^{n+1}$$

3-rd step

u^{n+1} must fulfill the continuity!
Let's take the numerical divergence:

$$\tilde{\Delta} P^{n+1} = A_P \tilde{\nabla} \cdot \breve{u}_i \rightarrow P^{n+1}$$

2-nd step

Due to using old values for the neighboring velocities in the 3-rd step this is not fully accurate. We need to iterate.

P-u iteration for steady flow (2)

- **Inner iteration**:
  Iterative solution methods are used for solving the algebraic systems in 1-st and in 2-nd step. Unusually only 1 inner iteration step is done.
- **Pressure equation:**
  The Poisson equation is solved for pressure correction (not for pressure). This reduces the round-off error.
- **SIMPLE, SIMPLEC, SIMPLER, PISO**
- **Time dependent models:**
  When modeling transient flows we can include the time derivatives into Q. This way, the application of implicit integration scheme is possible, which allows much larger time steps.