Compressible Flows

Dr. Gergely Kristóf

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Explicit numerical schemes for compressible flows

- We can assume, that the state of a computational element is determined by its first neighbors.
- That way, the solution of large algebraic systems can be avoided.
- The price to be paid: acoustic waves need to be resolved, that is, the time step size is limited.

1D isentropic flows

Unsteady isentropic flow in a constant cross-section pipe.
Eg. in an exhaust pipe.

Continuity:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0$$

Euler equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Isentropic relation:

$$\frac{p}{\rho^{\gamma}}=\frac{p_0}{\rho_0^{\gamma}}$$

p_0 and ρ_0 are the pressure and density in the reference state.
p, ρ, u are unknown functions of x and t.

Introduction of the sound speed "a" as a new field variable

Only one state variable can be chosen in isentropic system.
We can use the speed of sound "a" to express the pressure (p) and density (ρ).
Both "u" and "a" do have the dimension of m/s.

$$\frac{p}{\rho^{\gamma}}=\frac{p_0}{\rho_0^{\gamma}}$$

$$\ln(p)-\gamma\ln(\rho)=\ln\left(\frac{p_0}{\rho_0^{\gamma}}\right)$$

$$\frac{dp}{p}=\gamma\frac{d\rho}{\rho}$$

$$a^2=\left.\frac{\partial p}{\partial \rho}\right|_{s=áll.}=\gamma\frac{p}{\rho}=\gamma\,\rho^{\gamma-1}\frac{p}{\rho^{\gamma}}=\gamma\frac{p_0}{\rho_0^{\gamma}}\rho^{\gamma-1}$$

$$2\ln(a)=(\gamma-1)\ln(\rho)+\ln\left(\gamma\frac{p_0}{\rho_0^{\gamma}}\right)$$

$$2\frac{da}{a}=(\gamma-1)\frac{d\rho}{\rho}$$

$$\frac{\partial a}{\partial p}=\frac{\gamma-1}{2\gamma}\frac{a}{p}\qquad\qquad\frac{\partial a}{\partial \rho}=\frac{\gamma-1}{2}\frac{a}{\rho}$$

We reformulate the governing equations

Continuity:

$$\frac{\partial \rho}{\partial t}\frac{\partial a}{\partial \rho}+u\frac{\partial \rho}{\partial x}\frac{\partial a}{\partial \rho}+\rho\frac{\partial u}{\partial x}\frac{\gamma-1}{2}\frac{a}{\rho}=0$$

$$\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial x}+\frac{\gamma-1}{2}a\frac{\partial u}{\partial x}=0 \qquad (1)$$

Euler equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}\underbrace{\frac{\partial a}{\partial p}\frac{2\gamma}{\gamma-1}\frac{p}{a}}_{1}=0$$

a²

$$\frac{\gamma-1}{2}\frac{\partial u}{\partial t}+\frac{\gamma-1}{2}u\frac{\partial u}{\partial x}+a\frac{\partial a}{\partial x}=0 \qquad (2)$$

$$\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial x}+\frac{\gamma-1}{2}a\frac{\partial u}{\partial x}=0 \qquad (1)$$

$$\frac{\gamma-1}{2}\frac{\partial u}{\partial t}+\frac{\gamma-1}{2}u\frac{\partial u}{\partial x}+a\frac{\partial a}{\partial x}=0 \qquad (2)$$

(1) + (2)

$$\frac{\partial}{\partial t}\underbrace{\left(a+\frac{\gamma-1}{2}u\right)}_{\alpha}+(u+a)\frac{\partial}{\partial x}\underbrace{\left(a+\frac{\gamma-1}{2}u\right)}_{\alpha}=0$$

$$\frac{\partial \alpha}{\partial t}+(u+a)\frac{\partial \alpha}{\partial x}=0$$

α=const. in the direction of C_+=dx/dt=u+a.

(1) - (2)

$$\frac{\partial}{\partial t}\underbrace{\left(a-\frac{\gamma-1}{2}u\right)}_{\beta}+(u-a)\frac{\partial}{\partial x}\underbrace{\left(a-\frac{\gamma-1}{2}u\right)}_{\beta}=0$$

$$\frac{\partial \beta}{\partial t}+(u-a)\frac{\partial \beta}{\partial x}=0$$

β=const. in the direction of C_-=dx/dt=u-a

Characteristics

C_+ and C_- are the characteristic directions. α and β are Riemann invariants.

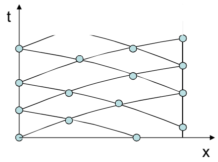
u and a can be expressed in terms of α and β.

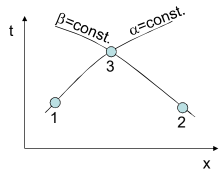
$$\alpha = a + \frac{\gamma-1}{2}u \qquad \beta = a - \frac{\gamma-1}{2}u \quad \rightarrow \quad a = \frac{\alpha+\beta}{2} \qquad u = \frac{\alpha-\beta}{\gamma-1}$$

Every field variable can than be expressed in terms of a:

$$\left(\frac{a}{a_0}\right)^2 = \frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

Numerical solution





$$\alpha_3 = \alpha_1 \qquad \beta_3 = \beta_2 \quad \rightarrow \quad a_3 = \frac{\alpha_3+\beta_3}{2} \qquad u_3 = \frac{\alpha_3-\beta_3}{\gamma-1}$$

C_+=dx/dt=u+a $\quad x_3 - x_1 = 0.5\left[(u_3+a_3)+(u_1+a_1)\right](t_3-t_1)+o(\Delta t^2)$

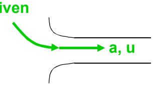
C_-=dx/dt=u-a $\quad x_3 - x_2 = 0.5\left[(u_3-a_3)+(u_2-a_2)\right](t_3-t_2)+o(\Delta t^2)$

t_3, x_3 can be calculated.

Boundary conditions

Inflow:

T_0, p_0, a_0 are given



the energy equation

$$T_0 = T + \frac{u^2}{2c_p} = \frac{a^2}{\gamma R} + \frac{u^2}{2c_p}$$

$$T_0 = \frac{1}{\gamma R}\left(\frac{\alpha+\beta}{2}\right)^2 + \frac{1}{2c_p}\left(\frac{\alpha-\beta}{\gamma-1}\right)^2$$

Either α or β is already given. (Along the outrunning characteristic curve.)
The other quantity can be expressed from the above equation.

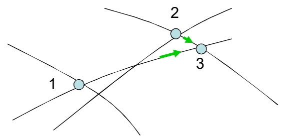
Outflow:

$$a_0 = a = \frac{\alpha+\beta}{2}$$

Closed pipe: $u = 0 \rightarrow \frac{\alpha-\beta}{\gamma-1} = 0 \rightarrow \alpha = \beta$

The problems...

- The numerical resolution depend on the actual physical properties, therefore it can become very coarse in some regions.
- The characteristic curves running in the same direction can intersect each other.



Finite volume method

The density based approach.

Continuity: $\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} = 0$

Eq.of motion: $\frac{\partial\rho u}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} = 0$

Energy eq.: $\frac{\partial\rho e}{\partial t} + \frac{\partial(\rho u e + p u)}{\partial x} = 0$

Equation of state:

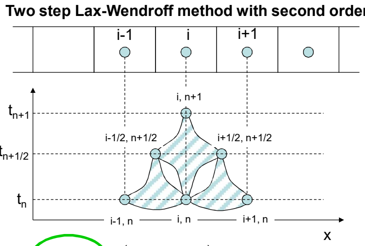
$p = \rho R T$

$e = c_v T + \frac{u^2}{2}$

In vector format: $\frac{\partial \underline{U}}{\partial t} + \frac{\partial \underline{F}}{\partial x} = \underline{Q}$

$$\underline{U} = \begin{bmatrix}\rho \\ \rho u \\ \rho e\end{bmatrix} \qquad \underline{F} = \begin{bmatrix}\rho u \\ \rho u^2 + p \\ \rho u e + p u\end{bmatrix} \qquad \underline{Q} = \begin{bmatrix}0 \\ 0 \\ 0\end{bmatrix}$$

Two step Lax-Wendroff method with second order accuracy:



Step 1:

$$\frac{U_{i+1/2}^{n+1/2} - (U_i^n + U_{i+1}^n)/2}{\Delta t/2} + \frac{F_{i+1}^n - F_i^n}{\Delta x} = \frac{Q_i^n + Q_{i+1}^n}{2}$$

When U is known ρ, u and e can be calculated. Eg. ρ=(ρu)/u
p is than obtained from the equation of state.
F and Q values can than be calculated at the time level n+1/2.

Step 2:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n+1/2}}{\Delta x} = \frac{Q_{i-1/2}^{n+1/2} + Q_{i+1/2}^{n+1/2}}{2}$$

This is an explicit time marching scheme. Only conditionally stable.
According to the linear stability theory:

$$\Delta t = \sigma \frac{\Delta x}{a + |u|} \qquad \sigma \leq 1 \quad \text{Courant number}$$

Strong oscillations can take place in the presence of shockwaves.
Fluxes must be corrected by using some upwinding or artificial viscosity.

A similar approach in FLUENT: density based solver + explicit formulation (time integration). The multi step time integration method implemented in FLUENT allows somewhat larger Courant number. (The default value is σ=1.)

Specification of the boundary conditions:
the method of characteristics can be used at the domain boundaries.
(There are other approaches too.)