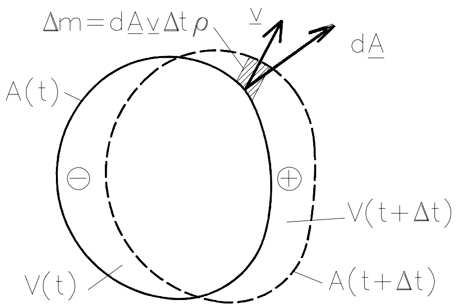
21. Integral momentum equation

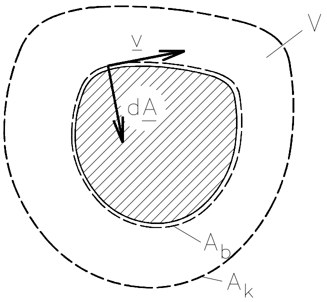
$$\frac{d}{dt}\int_{V(t)} \rho \underline{v} dV = \int_{V(t)} \rho \underline{g} dV - \int_{A(t)} p d\underline{A}$$ $\boxed{\mu = 0}$



$$\frac{d}{dt}\int_{V(t)} \rho \underline{v} dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} (\rho \underline{v})_{t+\Delta t} dV - \int_{V(t)} (\rho \underline{v})_t dV\right]$$

$$\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_A \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p d\underline{A}$$ where V is the control volume

<u>Solid body in control volume</u>



$$\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_{A_k} \underline{v}\rho(\underline{v} d\underline{A}) + \int_{A_b} \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_{A_k} p d\underline{A} - \int_{A_b} p d\underline{A}$$

$$\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_A \underline{v}\rho(\underline{v}\, d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p\, d\underline{A} - \underline{R}$$

$\underline{R}$[N] force acting on the body in the control volume

In case of steady flow:

$$\int_A \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p d\underline{A} - \underline{R}$$

$$\boxed{\frac{\partial}{\partial t}\int_V (\rho \underline{v}) dV + \int_A \underline{v}\rho(\underline{v} d\underline{A}) = \int_V \rho \underline{g} dV - \int_A p d\underline{A} - \underline{R} + \underline{S}}$$

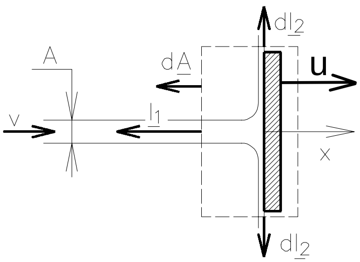
where $\underline{S}$ is the friction force.

Moment-of-momentum equation

$$\frac{\partial}{\partial t}\int_V \underline{r}\times(\rho\underline{v})dV+\int_A \underline{r}\times\underline{v}\rho(\underline{v}d\underline{A})=\int_V \underline{r}\times\rho\underline{g}dV-\int_A \underline{r}\times p d\underline{A}-\underline{M}+\underline{M}_s$$

22. Application of integral momentum equation

Stationary and moving deflector



1. Taking up the control volume: solid body is in the control volume, surfaces are ⊥ or || to the velocity vector.

$$\underset{\mathrm{I}}{\frac{\partial}{\partial t}\int_V (\rho\underline{v})dV}+\underset{\mathrm{II}}{\int_A \underline{v}\rho(\underline{v}d\underline{A})}=\underset{\mathrm{III}}{\int_V \rho\underline{g}dV}-\underset{\mathrm{IV}}{\int_A p d\underline{A}}-\underset{\mathrm{V}}{\underline{R}}+\underset{\mathrm{VI}}{\underline{S}}$$

2. Simplification of equation

$$\int_A \underline{v}\rho(\underline{v}d\underline{A})=-\underline{R}$$

3. Evaluation of integrals

$$\underline{I}_1=\int_{A_1}\underline{v}\rho(\underline{v}d\underline{A})=\rho v_1^2 A_1\left(-\underline{v}_1/|\underline{v}|_1\right),$$

$$d\underline{I}_2=\rho v_2^2 dA_2\frac{\underline{v}_2}{|\underline{v}|_2}$$

$$|\underline{I}|=\rho v^2 A\,.$$

4. Balances of forces in co-ordinate directions

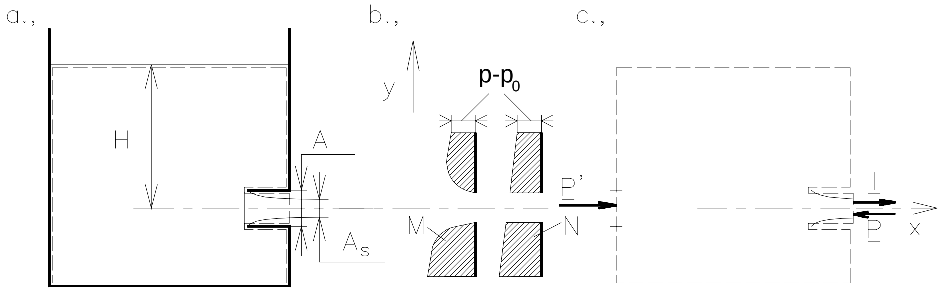
$$-I_1=-R_x \text{ azaz} -\rho v_1^2 A_1=-R_x=-R$$

$$R=\rho v_1^2 A_1$$

If the deflector moves: $\underline{u}>0$

instead of $\underline{v}_1$ $\underline{w}_1=\underline{v}_1-\underline{u}\Rightarrow R=\rho(v_1-u)^2 A_1$.

Borda discharge nozzle, contraction of jet

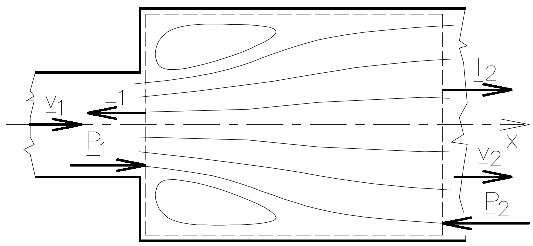


$$\alpha = A_S / A \qquad v = \sqrt{2gH}$$

$$I = P' - P \;\Rightarrow\; \rho v^2 A_s = (p_0 + \rho g H)A - p_0 A$$

$$2\rho g H A_s = \rho g H A \;\Rightarrow\; \alpha = A_s / A = 0.5$$

Change of pressure in Borda-Carnot sudden enlargement



$$-I_1 + I_2 = P_1 - P_2\,, \quad -\rho v_1^2 A_1 + \rho v_2^2 A_2 = -p_2 A_2 + p_1 A_2\,.$$

$$\rho v_1 A_1 = \rho v_2 A_2\,, \quad (p_2 - p_1)_{BC} = \rho v_2 (v_1 - v_2)\,, \quad (p_2 - p_1)_{id} = \frac{\rho}{2}\left(v_1^2 - v_2^2\right)$$

$$\Delta p'_{BC} = (p_2 - p_1)_{id} - (p_2 - p_1)_{BC} = \frac{\rho}{2}\left(v_1^2 - v_2^2\right) - \rho v_2 (v_1 - v_2)$$

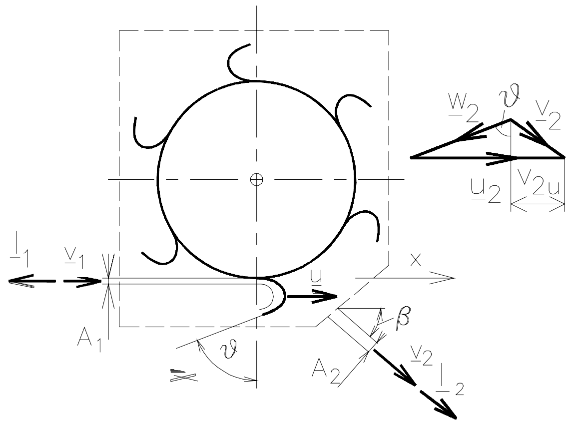
Borda-Carnot loss

$$\Delta p'_{BC} = \frac{\rho}{2}(v_1 - v_2)^2$$

Force acting on Borda-Carnot sudden enlargement

$$\underline{R} = (p_0 - p_1)(A_2 - A_1) = \rho v_2 (v_1 - v_2)(A_2 - A_1)$$

A Pelton-wheel



$$-I_1 + I_{2u} = -R_u, \quad I_1 = \rho v_1^2 A_1, I_{2u} = \rho v_2^2 A_2 \cos\beta, \ \rho v_1 A_1 = \rho v_2 A_2$$

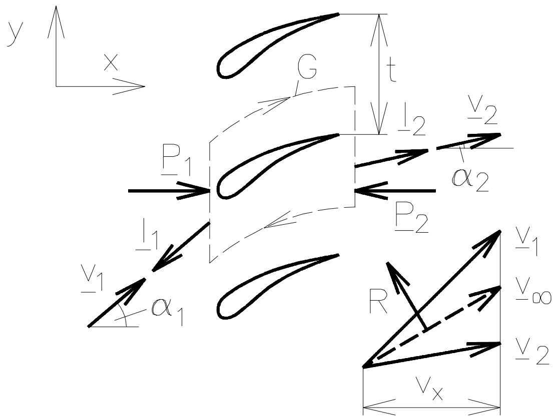
$$R_u = \rho v_1 A_1 (v_1 - v_{2u}), \text{ where } v_{2u} = v_2 \cos\beta$$

$$v_{2u} = u - w_2 \sin\vartheta, \ w_2 = w_1, \ w_1 = v_1 - u, \quad R_u = \rho v_1 A_1 (v_1 - u)(1 + \sin\vartheta)$$

$$R_u = \rho v_1 A_1 (v_1 - u)(1 + \sin\vartheta), \ \vartheta = 90° \ \frac{\partial P}{\partial u} = 2\rho v_1 A_1 [(v_1 - u) - u] = 0, \ u = v_1/2$$

$$P_{max} = \rho v_1 A_1 v_1^2 / 2$$

Force acting on one element of a infinite blade row, Kutta-Joukowsky theorem



$$-I_{1x} + I_{2x} = P_1 - P_2 - R_x, \quad -I_{1y} + I_{2y} = -R_y$$

$$I_1 = \rho v_x t v_1, \ I_2 = \rho v_x t v_2, \ P_1 = p_1 t \quad P_2 = p_2 t,$$

$$R_x = (p_1 - p_2)t + \rho v_x t (v_1 \cos\alpha_1 - v_2 \cos\alpha_2)$$

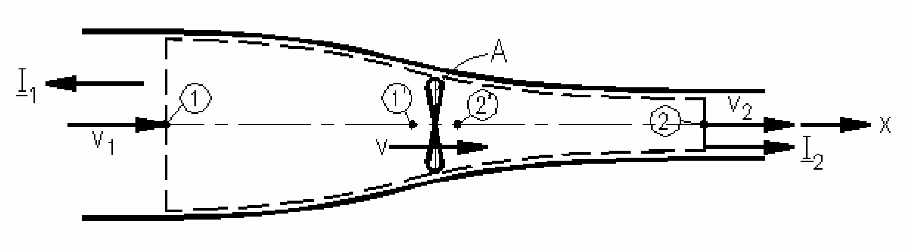
$$v_{1x} = v_{2x} \Rightarrow p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2) = \frac{\rho}{2}(v_{2y}^2 - v_{1y}^2) \Rightarrow R_x = \frac{\rho}{2}(v_{2y} - v_{1y}) t (v_{2y} + v_{1y})$$

$$R_y = \rho v_x t (v_{1y} - v_{2y}). \ \Gamma = (v_{1y} - v_{2y}) t \Rightarrow R_x = -\rho\Gamma \frac{v_{1y} + v_{2y}}{2}, R_y = \rho\Gamma v_x,$$

$$|\underline{R}| = \sqrt{R_x^2 + R_y^2} = \rho\Gamma \sqrt{v_x^2 + \left(\frac{v_{1y} + v_{2y}}{2}\right)^2} \Rightarrow |\underline{R}| = \rho |\underline{v}_\infty| \Gamma \ [N/m]$$

$$t \to \infty, \ v_{1y} - v_{2y} \to 0 \ \Gamma = t(v_{1y} - v_{2y}) = \text{áll.} \ \underline{v}_2 \to \underline{v}_1 \to \underline{v}_\infty \ |\underline{R}| = \rho |\underline{v}_\infty| \Gamma \ [N/m]$$

Propellers (actuator disc)



v_1 = u, velocity $v_1 \Rightarrow v_2$

$$\frac{\partial}{\partial t}\int_V (\rho \underline{v})dV + \int_A \underline{v}\rho(\underline{v}d\underline{A}) = \int_V \rho \underline{g}dV - \int_A pd\underline{A} - \underline{R} + \underline{S}$$

I II III IV V VI

$$-I_1 + I_2 = -R_x \Rightarrow R_x = \rho v_1^2 A_1 - \rho v_2^2 A_2$$

By using continuity equation: $R_x = \rho v A(v_1 - v_2)$

Bernoulli-equations 1 - 1' and 2'- 2

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} = \frac{v_{1'}^2}{2} + \frac{p_{1'}}{\rho} \qquad \frac{v_{2'}^2}{2} + \frac{p_{2'}}{\rho} = \frac{v_2^2}{2} + \frac{p_2}{\rho}$$

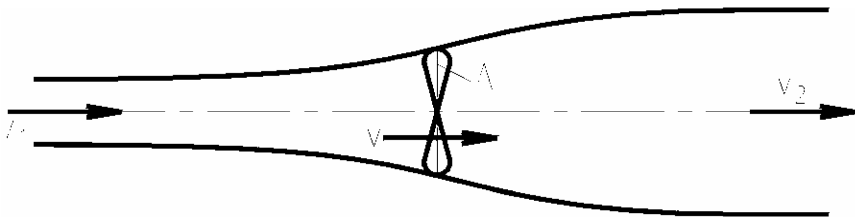
Since $p_1 = p_2$ and $v_{1'} = v_{2'} = v$, $p_{1'} - p_{2'} = \frac{\rho}{2}\left(v_1^2 - v_2^2\right)$

$$R_x = \frac{\rho}{2}(v_1^2 - v_2^2)A = \rho v A(v_1 - v_2),\ v = \frac{v_1 + v_2}{2} \Rightarrow R_x = \frac{\rho}{2}(v_1^2 - v_2^2)A$$

$P_u = v_1 R$ (useful power) $P_i = v\,R$ (input power) theoretical propellor efficiency:

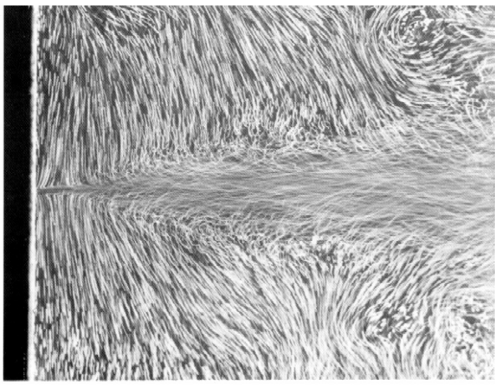
$$\eta_p = \frac{v_1 R}{vR} = \frac{v_1}{v} = \frac{2}{1 + v_2/v_1}$$

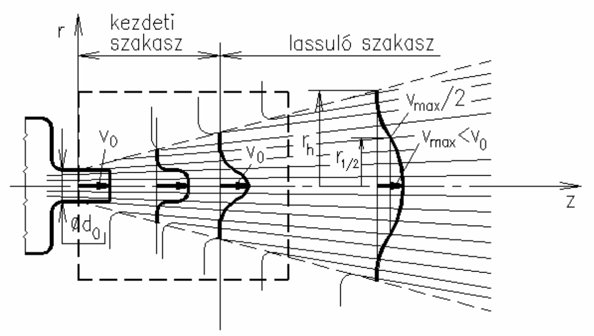
Wind turbine



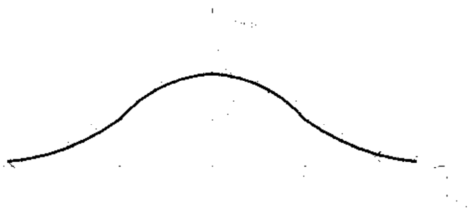
$$P = \rho A \frac{v_1 + v_2}{2}(v_1^2 - v_2^2) \Rightarrow \ \partial P/\partial v_2 = 0 \Rightarrow v_2 = \frac{1}{3}v_1 \Rightarrow \ P = \frac{16}{27}\rho A v_1^3$$

Jet





kezd.sz.:initial section, lassuló sz.: deceleration section

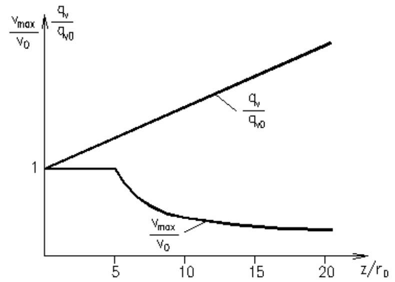


$r_h \sim z$, $r_{1/2} = k_1 z$,
$r_{1/2}/r_0 = k_1 z/r_0$

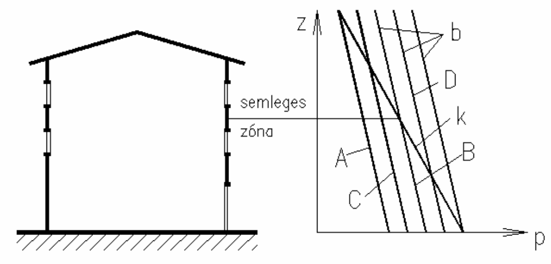
$$\int_A \underline{v}\rho\left(\underline{v}d\underline{A}\right)=0 \quad \Rightarrow \rho v_0^2 r_0^2 \pi = \int_A \rho v^2\, 2r\pi\, dr \Rightarrow v_0^2 r_0^2 = v_{max}^2 r_{1/2}^2\, 2 \int_0^{r_h/r_{1/2}} \frac{v^2}{v_{max}^2}\frac{r}{r_{1/2}}\, d\frac{r}{r_{1/2}}$$

$$\frac{v_{max}}{v_0} = \frac{Konst.}{\frac{r_{1/2}}{r_0}} \Rightarrow \frac{v_{max}}{v_0} = \frac{Konst.}{\frac{z}{r_0}}$$

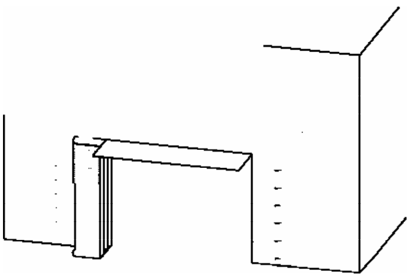
$$q_v = \int_0^{r_h} 2r\pi v\, dr = v_{max} r_{1/2}^2\, 2\pi \int_0^{r_h/r_{1/2}} \frac{v}{v_{max}}\frac{r}{r_{1/2}}\, d\frac{r}{r_{1/2}} \Rightarrow \frac{q_v}{q_{v0}} = Konst\frac{v_{max}}{v_0}\frac{r_{1/2}^2}{r_0^2} \quad \Rightarrow \frac{q_v}{q_{v0}} = Konst.\frac{z}{r_0}$$

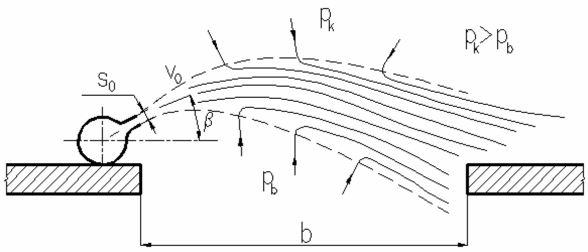


Air curtains

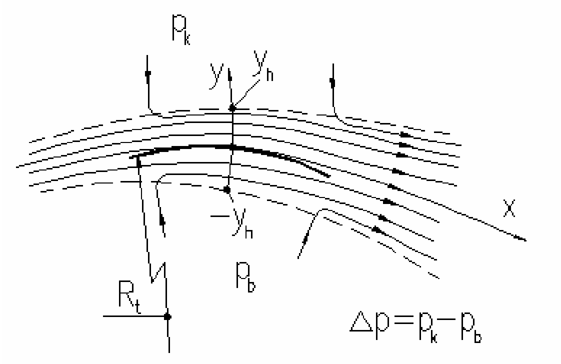


Neutral height





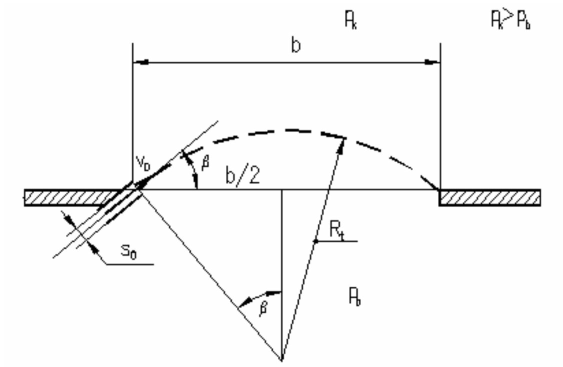
a.



b.

$$\frac{v^2}{R}=\frac{1}{\rho}\frac{\partial p}{\partial n}\Rightarrow\int_{p_b}^{p_k}dp=\int_{-y_h}^{y_h}\rho\frac{v^2}{R}dy\Rightarrow\ \Delta p=p_k-p_b=\frac{1}{R}\int_{-y_h}^{y_h}\rho v^2dy\Rightarrow\ \rho v_0^2 s_0=\int_{-y_h}^{y_h}\rho v^2dy$$

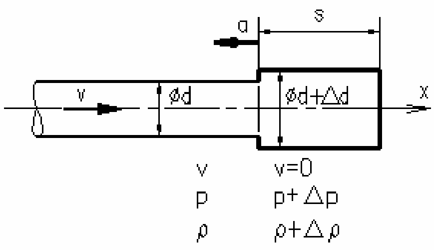
$$\Delta p=\frac{\rho v_0^2 s_0}{R}\Rightarrow R=\frac{\rho v_0^2 s_0}{\Delta p},$$

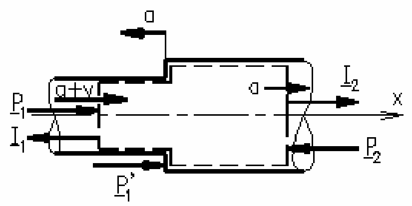


b=2 R_t sinβ. $b=2\frac{\rho v_0^2 s_0}{\Delta p}\sin\beta$, $B=\frac{b}{s_0}$, $D=\frac{\Delta p}{\frac{\rho}{2}v_0^2}$, $B=\frac{K}{D}\sin\beta$

theoretical: K=4, experimental: K=1.71+0.0264 B ($25\le\beta^0\le45$ and $10\le B\le40$)

<u>Allievi theorem</u>





pipe diameter d ⇒ d+Δd

Shortening of water column. Compression of water: Δs_1 and expansion of pipe wall: Δs_2 $\Rightarrow \Delta s = \Delta s_1 + \Delta s_2$

$\Delta s_1 = \frac{\Delta p}{E_v} s$, $E_v = 2.1 \cdot 10^9$Pa, $\Delta s_2 \frac{d^2 \pi}{4} = s d\pi \frac{\Delta d}{2}$ $\Delta\sigma = \frac{\Delta p\, d}{2\delta}$; $\frac{\Delta d}{d} = \frac{\Delta \sigma}{E_a} = \frac{\Delta p\, d}{2\delta E_a}$

$E_{st} = 2 \cdot 10^{11}$Pa $\Rightarrow \Delta s_2 = \frac{\Delta p\, d}{\delta E_a} s \Rightarrow \frac{\Delta s}{s} = \frac{\Delta s_1}{s} + \frac{\Delta s_2}{s} = \Delta p \left(\frac{1}{E_v} + \frac{d}{\delta E_a} \right) = \frac{\Delta p}{E_r}$

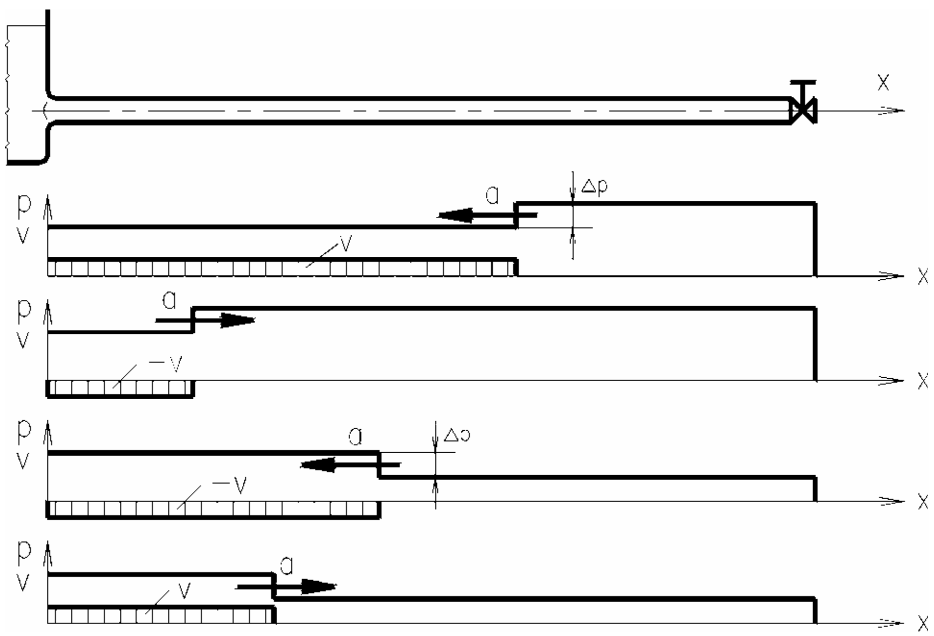
Integral momentum equation:

$$-\rho(a+v)A(a+v) + (\rho+\Delta\rho)\, a\, (A+\Delta A)a =$$
$$= pA + (p+\Delta p)\Delta A - (p+\Delta p)(A+\Delta A),$$

Continuity: $\rho(a+v)A = (\rho+\Delta\rho)a(A+\Delta A)$

$\Delta p = \rho v(a+v)$ v<<a $\Delta p = \rho v a$ $T_t \le \frac{2L}{a}$ turning off time

$s = at \Rightarrow A\,\Delta s = A\frac{\Delta p}{E_r} s = A\frac{\Delta p}{E_r} at \Rightarrow \frac{\Delta p}{E_r} a = v$ $a = \sqrt{\frac{E_r}{\rho}}$.



23. Flow of viscous fluids, Navier-Stokes equation

Rheology

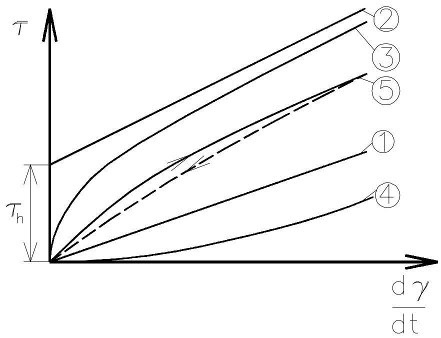
Shear stress versus strain rate (deformation rate)

curve 1: Newtonian fluid: $\underline{\tau_{yx} = \mu \frac{dv_x}{dy}} = \mu \frac{d\gamma}{dt}$.

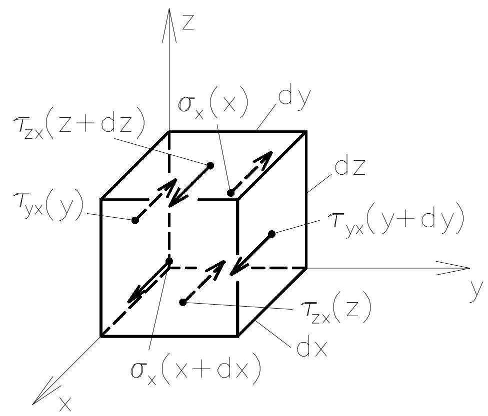
Non-Newtonian fluids:

curve 2: $\tau = \tau_h + \mu_\infty \frac{d\gamma}{dt}$,

Curves 3 and 4: $\tau = k(d\gamma / dt)^n$.



Differential momentum equation $\frac{d\underline{v}}{dt} = \underline{g} + \underline{F}$, In case of inviscid fluid: $\underline{F} = -\frac{1}{\rho}\mathrm{grad}p$



$$F_x = \frac{1}{\rho dx dy dz}\{[\sigma_x(x+dx) - \sigma_x(x)]dydz + [\tau_{yx}(y+dy) - \tau_{yx}(y)]dxdz$$
$$+ [\tau_{zx}(z+dz) - \tau_{zx}(z)]dxdy\}$$

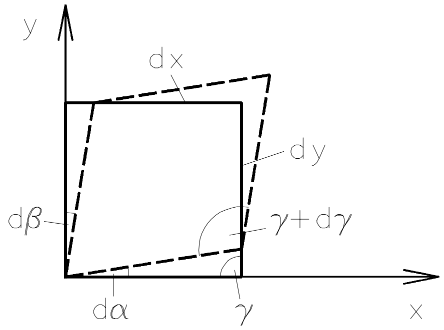
Since $\sigma_x(x+dx) = \sigma_x(x) + \frac{\partial \sigma_x}{\partial x}dx \Rightarrow F_x = \frac{1}{\rho}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$

$$\underline{\underline{\Phi}} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \qquad \underline{F} = \frac{1}{\rho}\underline{\underline{\Phi}}\underline{\nabla} \quad \Rightarrow \quad \underline{\nabla} = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$\boxed{\frac{d\underline{v}}{dt} = \underline{g} + \frac{1}{\rho}\underline{\underline{\Phi}}\underline{\nabla}}$$

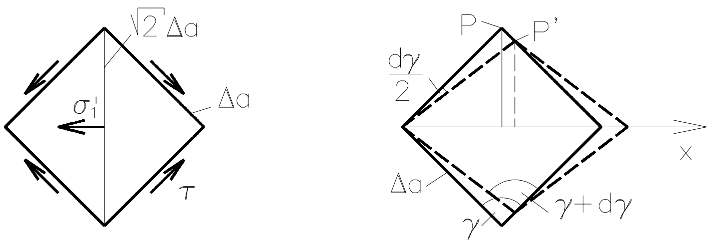
$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}=g_x+\frac{1}{\rho}\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)$$

Relation between stresses and deformation.



$$d\gamma = d\alpha + d\beta = \frac{\partial v_y}{\partial x}dt + \frac{\partial v_x}{\partial y}dt\ .$$

$$\tau_{xy}=\mu\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)=\tau_{yx}\ . \qquad p=-\frac{1}{3}\left(\sigma_x+\sigma_y+\sigma_z\right).$$



$$\sigma'_{1x}=\mu\frac{d\gamma}{dt}=2\mu\frac{\partial v_x}{\partial x}$$

$$\sigma_x=-p+2\mu\frac{\partial v_x}{\partial x}-\frac{2}{3}\mu \operatorname{div}\underline{v}\ .$$

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left[2\mu\frac{\partial v_x}{\partial x}-\frac{2}{3}\mu\operatorname{div}\underline{v}\right]+\right.$$

$$\left.+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\right]\right\}.$$

Navier-Stokes-equation

$\mu=\text{const.}$ and $\rho=\text{const.}$

$$\boxed{\begin{aligned}
\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}&=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)\\
\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}&=g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)\\
\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}&=g_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)
\end{aligned}}$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

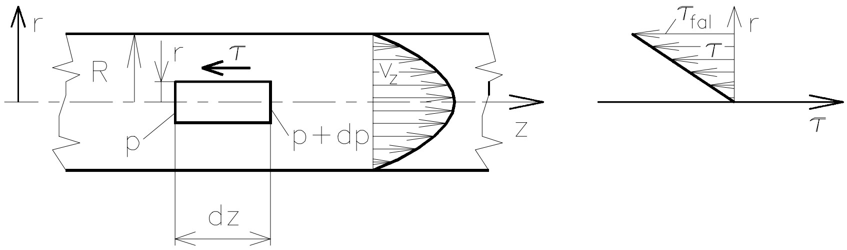
$$\boxed{\frac{d\underline{v}}{dt}=\frac{\partial \underline{v}}{\partial t}+\mathrm{grad}\frac{v^2}{2}-\underline{v}\times \mathrm{rot}\underline{v}=\underline{g}-\frac{1}{\rho}\mathrm{grad}p+\nu\Delta\underline{v}}$$ Navier-Stokes-equation

$\Delta\underline{v}=\mathrm{grad}\,\mathrm{div}\underline{v}-\mathrm{rot}\,\mathrm{rot}\underline{v}$.

Since $\mathrm{div}\underline{v}=0$ $\boxed{\frac{d\underline{v}}{dt}=\underline{g}-\frac{1}{\rho}\mathrm{grad}p-\nu\,\mathrm{rot}\,\mathrm{rot}\underline{v}}$ if rot rot $\underline{v}$ = 0, rot $\underline{v}$ = 0 viscosity exerts no effect.

24. Developed laminar pipe flow

Cylindrical symmetry of developed pipe flow: $v_r=0$, $\frac{\partial(\ )}{\partial z}=0$ (2D flow)



$$r^2\pi p-r^2\pi(p+dp)+2r\pi dz\tau=0. \Rightarrow\ 2\tau dz=rdp\ \Rightarrow\ \tau=\frac{1}{2}r\frac{dp}{dz}=\mu\frac{dv_z}{dr}$$

dp/dz= const.

$$\int dv_z=\frac{1}{2\mu}\frac{dp}{dz}\int rdr\Rightarrow v_z=\frac{1}{4\mu}\frac{dp}{dz}r^2+\text{áll.}$$

If r=R $\Rightarrow$ $v_z=0$ $\quad v_z=-\frac{1}{4\mu}\frac{dp}{dz}\left[R^2-r^2\right]\Rightarrow v_z>0$ if $\frac{dp}{dz}<0$

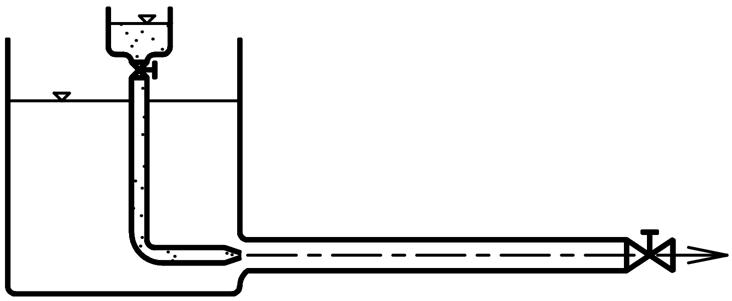
Friction loss over l: $\Delta p'[\mathrm{Pa}]\Rightarrow\frac{dp}{dz}=-\frac{\Delta p'}{l}\ \Rightarrow\ v_z=\frac{\Delta p'}{4\mu l}\left[R^2-r^2\right]\Rightarrow\tau=-\frac{\Delta p'}{2l}r$

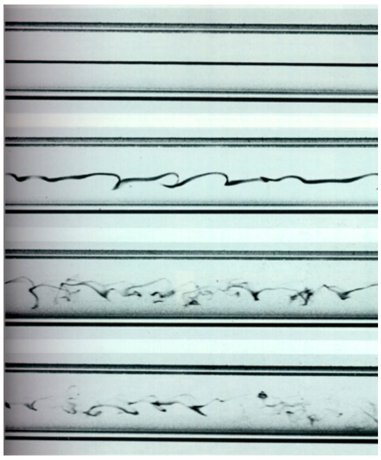
$$v_{z\max}=\frac{\Delta p' R^2}{4\mu l}\ \Rightarrow\ \bar{v}=\frac{v_{z\max}}{2}\ \Rightarrow\ \bar{v}=\frac{1}{8}\frac{\Delta p'}{\mu l}R^2$$

$\boxed{\Delta p'=\frac{8\mu\bar{v}l}{R^2}}$. Wall shear stress: $\tau_{wall}=-\frac{\Delta p' R}{2l}$ $\quad 2R\pi\, l\, \tau_{wall}=R^2\pi\Delta p'$

25. Laminar and turbulent flows

Properties of turbulent flow





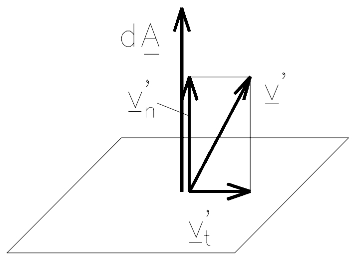
Transition depends on Reynolds number: $\mathrm{Re} = \dfrac{v d \rho}{\mu} \Rightarrow \mathrm{Re} \cong 2300$

time-average: , $\underline{\bar{v}} = \dfrac{1}{T}\int_0^T \underline{v}\,dt$ fluctuating velocity: $\underline{v}'$ $\left(\overline{\underline{v}'} = \dfrac{1}{T}\int_0^T \underline{v}'dt = 0\right)$

$\underline{v} = \underline{\bar{v}} + \underline{v}'$. $\quad p = \bar{p} + p'$.

Turbulence intensity: $Tu = \dfrac{\sqrt{\overline{(\underline{v}')^2}}}{|\underline{\bar{v}}|} = \dfrac{\sqrt{\overline{v_x'^2} + \overline{v_y'^2} + \overline{v_z'^2}}}{|\underline{\bar{v}}|}$

Apparent (Reynolds) stresses



$$\frac{\partial \bar{v}_x}{\partial t} + \frac{\partial\left(\bar{v}_x^2\right)}{\partial x} + \frac{\partial\left(\bar{v}_x \bar{v}_y\right)}{\partial y} + \frac{\partial\left(\bar{v}_x \bar{v}_z\right)}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\left(\frac{\partial^2 \bar{v}_x}{\partial x^2} + \frac{\partial^2 \bar{v}_x}{\partial y^2} + \frac{\partial^2 \bar{v}_x}{\partial z^2}\right) - \frac{\partial\left(\overline{v_x'^2}\right)}{\partial x} - \frac{\partial\left(\overline{v'_x v'_y}\right)}{\partial y} - \frac{\partial\left(\overline{v'_x v'_z}\right)}{\partial z}.$$

$$\int_A \underline{\bar{v}}\rho\underline{\bar{v}}d\underline{A} = -\int_A \bar{p}d\underline{A} + \int_V \rho\underline{g}dV - \int_A \overline{\underline{v}'\rho\underline{v}'}d\underline{A}$$

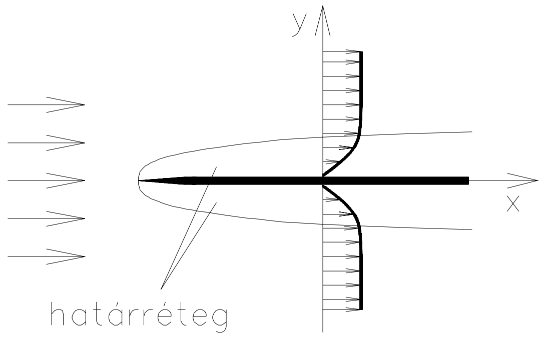
$$-\int_A \overline{\underline{v}'\rho\underline{v}'}d\underline{A} = -\int_A \overline{\underline{v}'_n\rho v'_n}dA - \int_A \overline{\underline{v}'_t\rho v'_n}dA\,.$$

$$\int_A \underline{\bar{v}}\rho\underline{\bar{v}}d\underline{A} = -\int_A \left[\bar{p} + \rho\left(\overline{v_n'^2}\right)\right]d\underline{A} + \int_V \rho\underline{g}dV - \int_A \rho\left(\overline{\underline{v}'_t v'_n}\right)d\underline{A}$$

$p_\ell = \rho\left(\overline{v_n'^2}\right)$ apparent pressure rise $\tau_\ell = -\rho\left(\overline{v'_t v'_n}\right)$ apparent shear stress

$$-\frac{\partial\left(\overline{v_x'^2}\right)}{\partial x} - \frac{\partial\left(\overline{v'_x v'_y}\right)}{\partial y} - \frac{\partial\left(\overline{v'_x v'_z}\right)}{\partial z} = \frac{1}{\rho}\left(\frac{\partial \sigma_{xt}}{\partial x} + \frac{\partial \tau_{yxt}}{\partial y} + \frac{\partial \tau_{zxt}}{\partial z}\right)$$

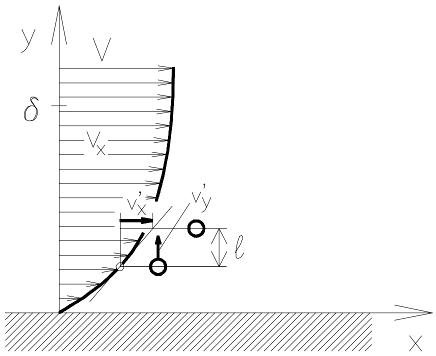
26. Boundary layers



$$v_z = 0, \frac{\partial(\ )}{\partial z} = 0, \ v_y << v_x \quad \frac{\partial(\ )}{\partial x} << \frac{\partial(\ )}{\partial y}$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = V \frac{dV}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$$



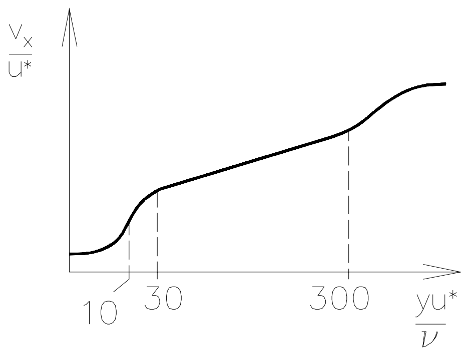
$$\tau = \tau_{yx\ell} = \rho \ell^2 \left| \frac{\partial v_x}{\partial y} \right| \frac{\partial v_x}{\partial y} = \mu_t \frac{\partial v_x}{\partial y}$$, 1 [m] mixing length

$$\mu_t = \rho \ell^2 \left| \frac{\partial v_x}{\partial y} \right|$$ eddy viscosity

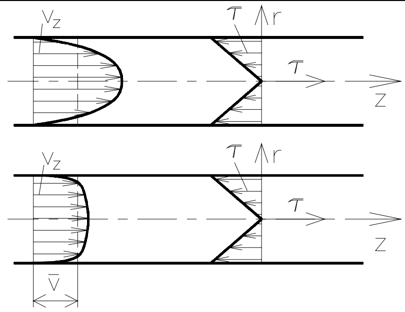
$$u^* = \sqrt{\frac{\tau_0}{\rho}}$$ [m/s] friction velocity

$$\boxed{\frac{v_x}{u^*} = \frac{1}{\kappa} \ln \frac{y u^*}{\nu} + K}$$ where $\kappa = 0.4$, $K = 5$

in viscous sublayer $\tau = \tau_0 = \mu \frac{\partial v_x}{\partial y}$, $\boxed{\frac{v_x}{u^*} = \frac{y u^*}{\nu}}$

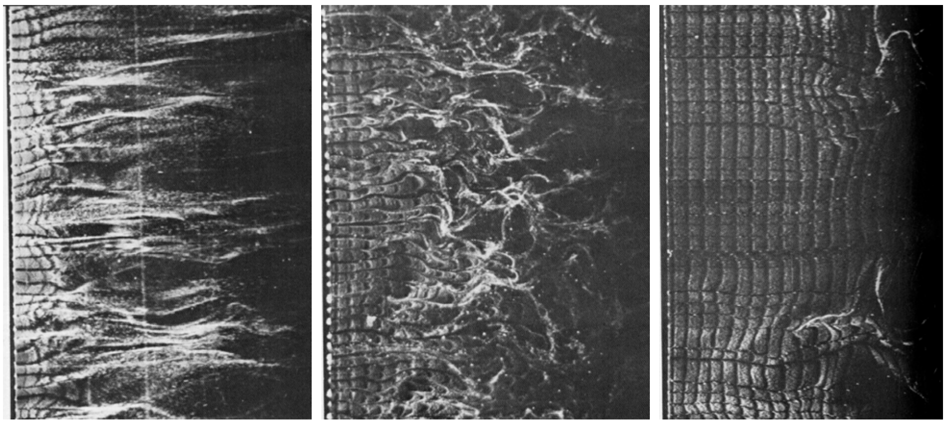


Velocity and shear stress distribution in laminar and turbulent pipe flows



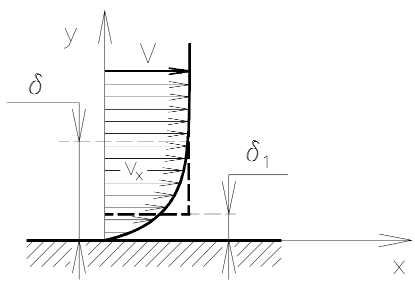
Characteristics of the flow in boundary layer

$$y^+ = \frac{yu^*}{\nu}$$



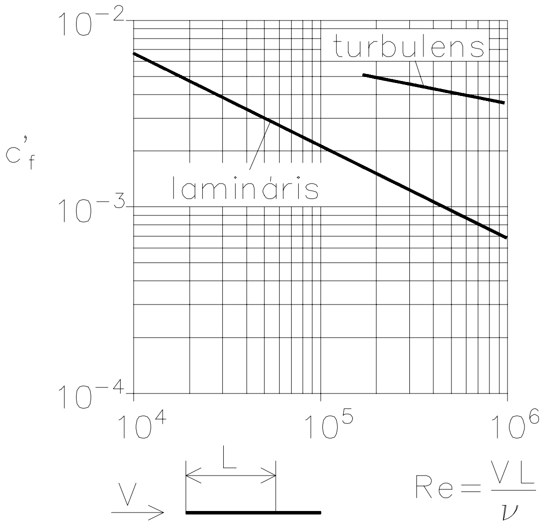
y^+=2,7 y^+=101 y^+=407

Displacement thickness

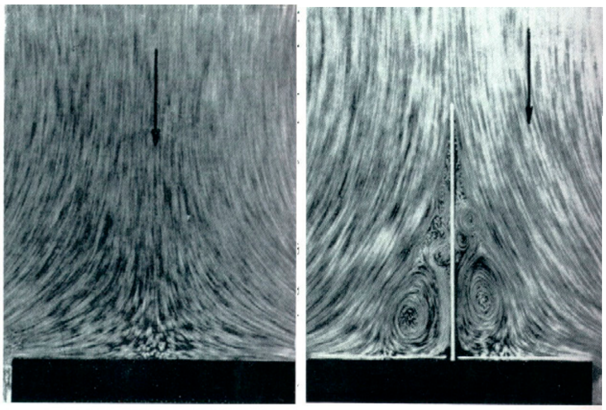


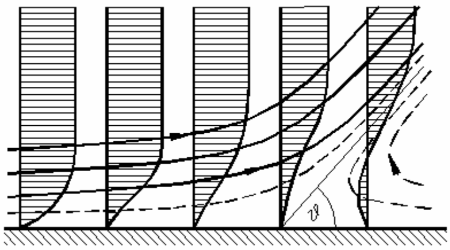
$$\delta_1 = \int_0^{\delta} \left(1 - \frac{v_x}{V}\right) dy \, .$$

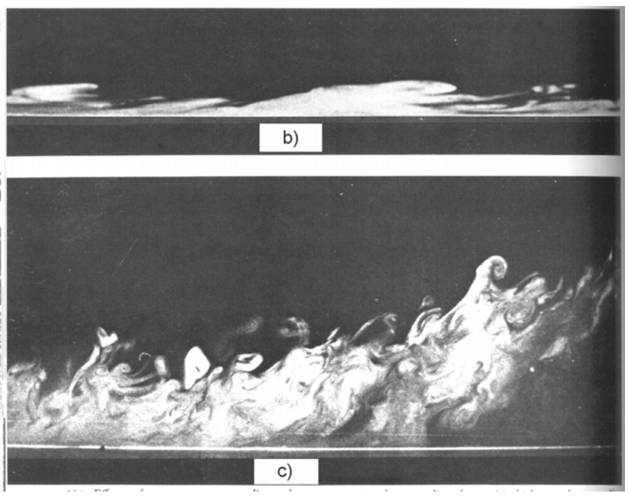
<u>Friction coefficient</u> $c'_f = \dfrac{\tau_0}{\dfrac{\rho}{2} V^2}$



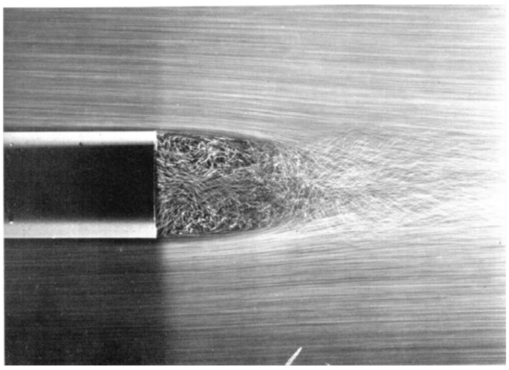
<u>Boundary layer separation</u>



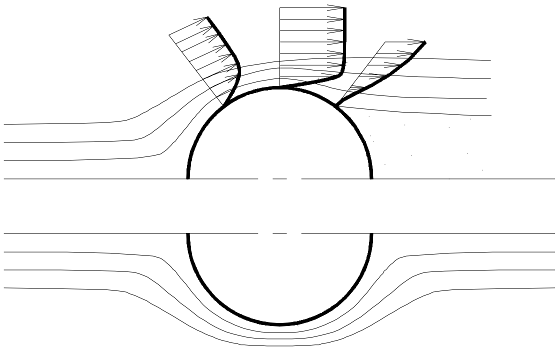

a)

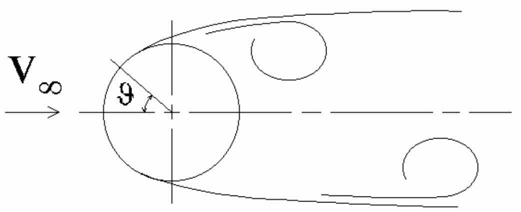

b)
c)

Separation bubble

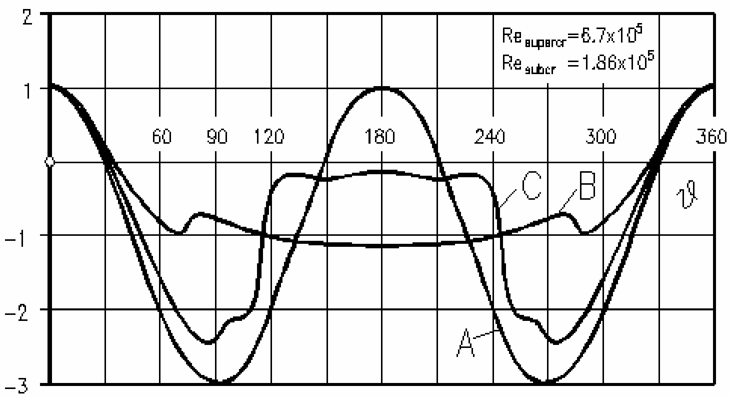


Flow past cylinders

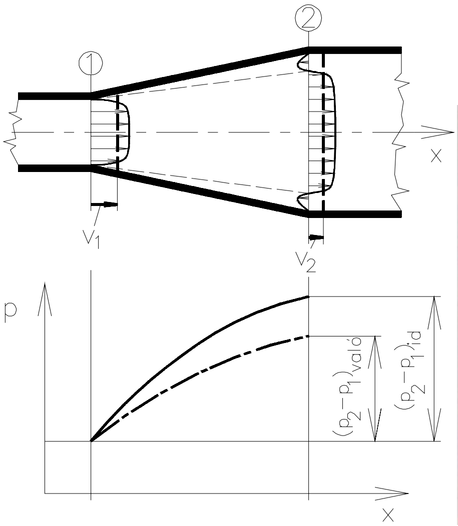


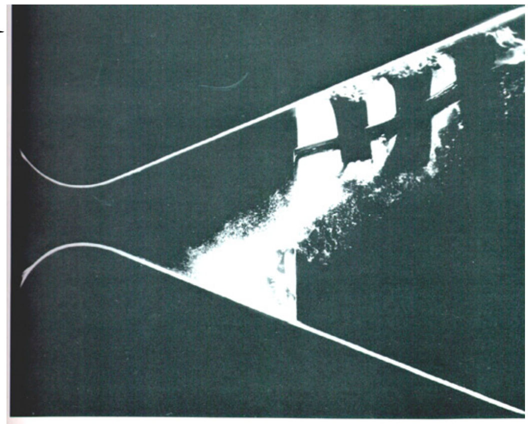


$$c_p = \frac{p - p_0}{\frac{\rho}{2} V^2}$$ pressure coefficient

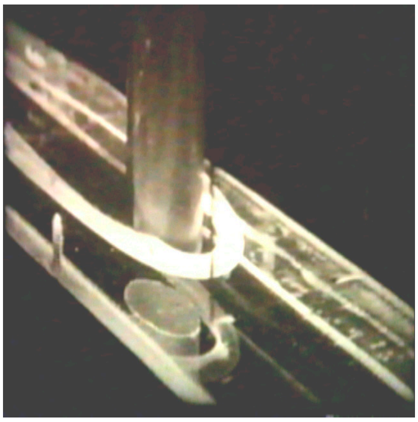


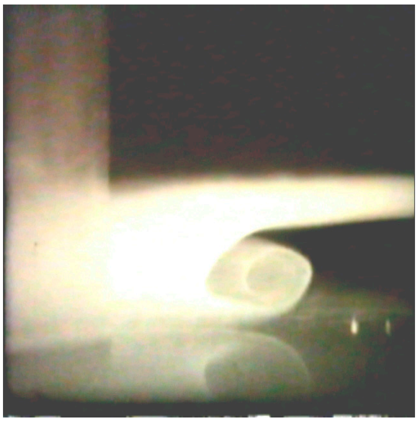
Flow in diffusers



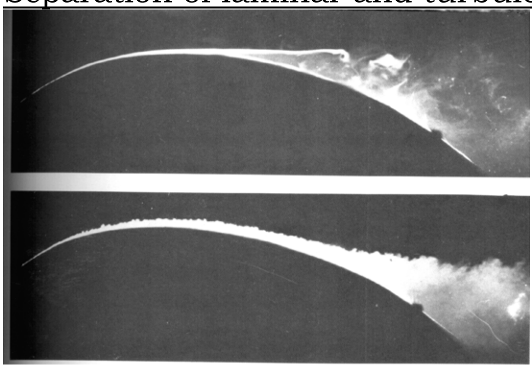


Horse-shoe vortex

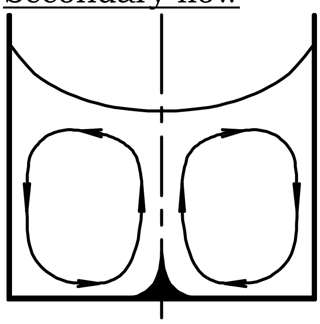




Separation of laminar and turbulent boundary layers

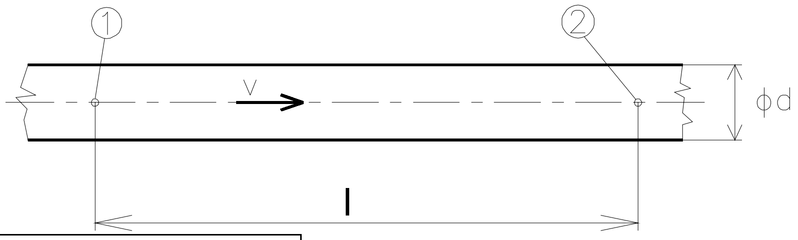


Secondary flow



27. Hidraulics

Bernoulli equation completed with friction losses



$$\boxed{\rho\frac{v_1^2}{2}+p_1+\rho U_1=\rho\frac{v_2^2}{2}+p_2+\rho U_2+\Delta p'}$$

Dimensional analyses

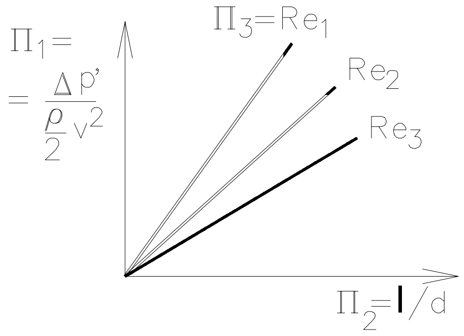
$$\Delta p' = f\left(\ell, \mu, \rho, d, v\right)$$

$[Q]=kg^{\alpha}m^{\beta}s^{\gamma}\Rightarrow Q_1,Q_2,......,Q_n.\Rightarrow F(Q_1,Q_2,......,Q_n)=0\Rightarrow \Pi_1,\Pi_2,\Pi_3,.....\Pi_{n-r}$, dimensionless parameters $\Rightarrow F(\Pi_1,\Pi_2,\Pi_3,.....\Pi_{n-r},)=0$:

$$\Pi=\Delta p'^{k_1}\ell^{k_2}\mu^{k_3}\rho^{k_4}d^{k_5}v^{k_6}.$$

$$\Pi_1=\frac{\Delta p'}{\frac{\rho}{2}v^2},\ \Pi_2=\ell/d,\ \Pi_3=Re=\frac{vd}{\nu}\ .$$

$$F(\Pi_1,\Pi_2,\Pi_3)=0$$



$$\Pi_1=f\left(\Pi_2,\Pi_3\right)$$

$$\frac{\Delta p'}{\frac{\rho}{2}v^2}=\lambda(Re)\frac{\ell}{d}\ \Rightarrow\ \boxed{\Delta p'=\frac{\rho}{2}v^2\frac{\ell}{d}\lambda(Re)}$$ where λ is the friction factor

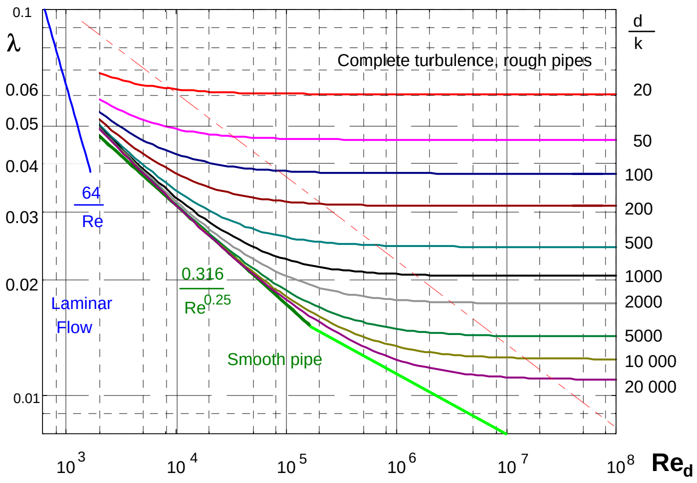
In case of laminar flow in pipe $\Delta p'=\frac{8\mu\ell}{R^2}v$, where $R=\frac{d}{2}\Rightarrow\Delta p'=\frac{\rho}{2}v^2\frac{\ell}{d}\frac{64\nu}{vd}$,

Since $\frac{vd}{\nu}=Re\Rightarrow\Delta p'=\frac{\rho}{2}v^2\frac{\ell}{d}\lambda_{lam}$ $\boxed{\lambda_{lam}=\frac{64}{Re}}$

<u>Rough wall</u> $k[m]$ diameter of sand particles, relative roughness $\Pi_4 = \frac{r}{k}$, where, $r=d/2$ pipe radius. $Re > 2300$ $Re \le Re_h$ $\frac{1}{\sqrt{\lambda_{turb}}} = 1.95\lg\left(Re\sqrt{\lambda_{turb}}\right) - 0.55$

$4000 \le Re \le 10^5$ Blasius formula $\boxed{\lambda_{turb} = \frac{0.316}{\sqrt[4]{Re}}}$

<u>Moody diagram</u> for determination of the friction factor λ for pipes



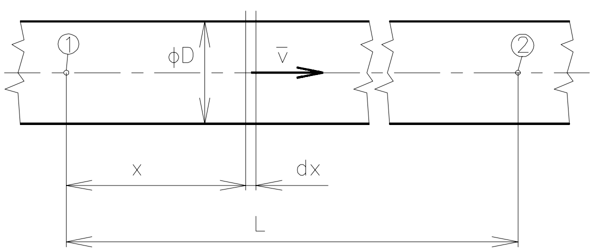
<u>Wall shear stress</u>

$$\Delta p' \frac{d^2 \pi}{4} = \frac{\rho}{2} v^2 \frac{\ell}{d} \lambda \frac{d^2 \pi}{4} = \tau_0 d \pi \ell \;\Rightarrow\; \tau_0 = \frac{\rho}{2} v^2 \frac{\lambda}{4}$$

<u>Pipes of noncircular cross sections</u>: $d_e = \frac{4A}{K}$, where $A[m^2]$ cross section, $K[m]$ wetted perimeter

$$\Delta p' = \frac{\rho}{2} v^2 \frac{\ell}{d_e} \lambda(Re) \text{ , where } Re = \frac{v d_e}{\nu}$$

<u>Compressible flow in pipe</u>

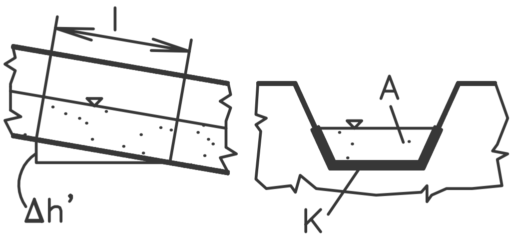


$$-dp = \frac{\rho}{2} v^2 \frac{dx}{D} \lambda \,,\, \bar{v} = \frac{q_m}{\rho A} \,,\, \rho = \frac{p}{RT} \;\Rightarrow\; -dp = \frac{q_m^2 RT}{2pA^2} \frac{\lambda}{D} dx \;\Rightarrow\; -\int_{p_1}^{p_2} p dp = \int_0^L \frac{q_m^2 RT\lambda}{2A^2 D} dx$$

Since $Re = \frac{\bar{v}D}{\nu} = \frac{q_m D}{\rho A \nu} = \frac{q_m D}{A\mu}$, $\mu = f(T)$ and $T \cong$ const. $\lambda \cong$ Const.

$$\frac{p_1^2-p_2^2}{2}=\frac{q_m^2 RT\lambda L}{2A^2 D}\frac{\rho_1^2}{\rho_1^2} \Rightarrow \frac{p_1^2-p_2^2}{2}=p_1\frac{\rho_1}{2}\overline{v}_1^2\frac{L}{D}\lambda \Rightarrow \boxed{\frac{p_1^2-p_2^2}{2}=p_1\Delta p'_{ink}}$$

Open channel flow



$\Delta h'=\frac{\overline{v}^2}{2g}\frac{\ell}{d_e}\lambda$, where $d_e=\frac{4A}{K}$, introducing $i=\frac{\Delta h'}{\ell}$ slope of the channel bottom

$\overline{v}=\sqrt{\frac{2gd_e}{\lambda}i}=C\sqrt{d_e i}$, where $C=\sqrt{\frac{2g}{\lambda}}$ Chézy equation, with $\lambda=0.02\sim0.03$ Chézy coefficient $C\cong 28$.

Losses in pipe components

Losses of development of pipe flow

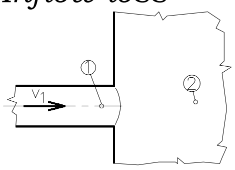
$$\Delta p'_{dev}=\frac{\rho}{2}v^2\zeta_{dev}$$

laminar flow: $\zeta_{dev,lam}\cong 1.2$, turbulent flow: $\zeta_{dev,torb}\cong 0.05$

Loss in Borda-Carnot extension

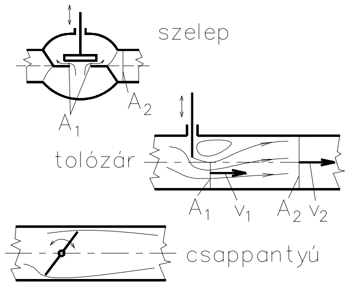
$$\Delta p'_{BC}=\frac{\rho}{2}(v_1-v_2)^2$$

Inflow loss



$$\Delta p'_{in}=\frac{\rho}{2}(v_1-0)^2=\frac{\rho}{2}v_1^2.$$

Losses in valves

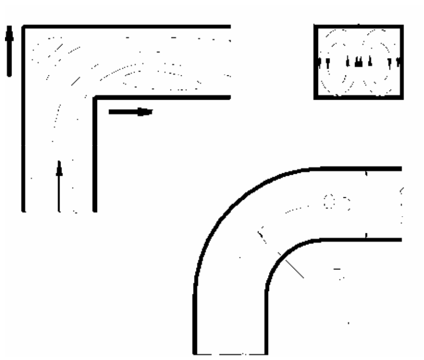


$$\Delta p'_v \cong \frac{\rho}{2} v_2^2 \left(\frac{v_1}{v_2} - 1 \right)^2, \zeta_v \cong \left(\frac{A_2}{A_1} - 1 \right)^2$$

Diffuser loss

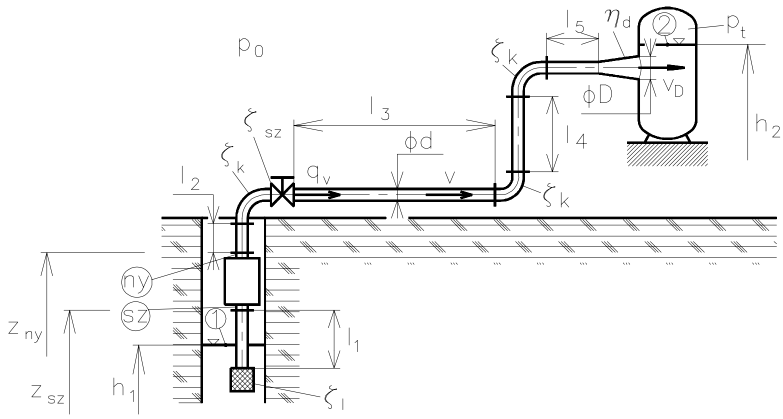
$\Delta p'_{diff} = (p_2 - p_1)_{id} - (p_2 - p_1)_{val} = (1 - \eta_d) \frac{\rho}{2} (v_1^2 - v_2^2)$ η_d diffuser efficiency

Losses in bends $\Delta p'_b = \frac{\rho}{2} v^2 \zeta_b$



Applications

Water supply system



Known parameters: d ℓ_1, ℓ_2 h_1, h_2 D $q_v \left[\frac{m^3}{s} \right]$, literature: $\zeta_l, \zeta_{sz}, \zeta_k$ η_d.

Calculation of pump head $H = \frac{\Delta p_{š}}{\rho g} + (z_{ny} - z_{sz})$ and performance $P_h = q_v \rho g H$

2 Bernoulli equations:

1) $\rho\frac{v_1^2}{2}+p_1+\rho U_1=\rho\frac{v_{sz}^2}{2}+p_{sz}+\rho U_{sz}+\Sigma\Delta p'_{sz}$, where $\sum\Delta p'_{sz}=\frac{\rho}{2}v^2\left(\zeta_1+\frac{\ell_1}{d}\lambda_1\right)$

2) $\rho\frac{v_{ny}^2}{2}+p_{ny}+\rho U_{ny}=\rho\frac{v_2^2}{2}+p_2+\rho U_2+\Sigma\Delta p'_{ny}$, where

$$\sum\Delta p'_{ny}=\frac{\rho}{2}v^2\left(\frac{\ell_2}{d}\lambda_2+\zeta_k+\zeta_{sz}+\frac{\ell_3}{d}\lambda_3+\zeta_k+\frac{\ell_4}{d}\lambda_4+\zeta_k+\frac{\ell_5}{d}\lambda_5\right)+$$

$$+(1-\eta_d)\frac{\rho}{2}\left(v^2-v_D^2\right)+\frac{\rho}{2}v_D^2.$$

Considering that $v_1=v_2=0$, $U_{ny}-U_{sz}=g(z_{ny}-z_{sz})$, $U_2-U_1=g(h_2-h_1)$, $p_1=p_0$, $p_2=p_t$ $\lambda_1=\lambda_2=....=\lambda$.

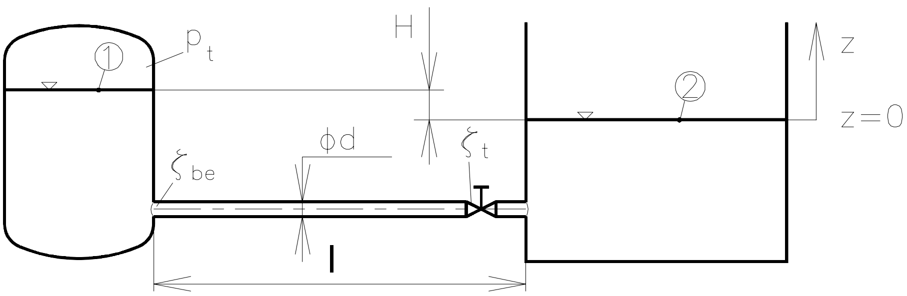
$$\Delta p_š=p_{nyš}-p_{szš}=p_t-p_0+\rho g(h_2-h_1)-\rho g(z_{ny}-z_{sz})+$$

$$+\frac{\rho}{2}v^2\left(\zeta_1+\zeta_{sz}+3\zeta_k+\frac{\Sigma\ell_i}{d}\lambda\right)+(1-\eta_d)\frac{\rho}{2}\left(v^2-v_D^2\right)+\frac{\rho}{2}v_D^2.$$

Continuity equation: $v d^2=v_D D^2$. $Re=\frac{vd}{\nu}\Rightarrow\lambda$

Flow in pipe connecting water tanks

Known parameters: d ℓ ζ_t. Calculation of $q_v\left[\frac{m^3}{s}\right]$



$$\rho\frac{v_1^2}{2}+p_1+\rho U_1=\rho\frac{v_2^2}{2}+p_2+\rho U_2+\Sigma\Delta p'$$

Considering $p_1=p_t$ and $p_2=p_0$, $U=gz$ és $z_2=0,\ z_1=H$, $v_1=v_2=0$ és a

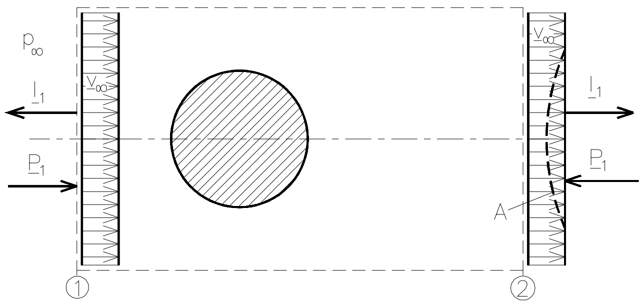
$$\Sigma\Delta p'_{sz}=\frac{\rho}{2}v^2\left(\zeta_{be}+\zeta_t+\frac{\ell}{d}\lambda+1\right)\Rightarrow p_t-p_0+\rho gH=\frac{\rho}{2}v^2\left(\zeta_{be}+\zeta_t+\frac{\ell}{d}\lambda+1\right)$$

$$v=\sqrt{\frac{2}{\rho}\frac{p_t-p_0+\rho gH}{(\zeta_{be}+\zeta_t+1)+\frac{\ell}{d}\lambda}}$$

$$v=\sqrt{\frac{A}{B+C\lambda}}$$

assuming $\lambda'\Rightarrow v'\Rightarrow Re'=\frac{v'd}{\nu}\Rightarrow\lambda''$–t .... $q_v=v\frac{d^2\pi}{4}$.

28. Aerodynamic forces and moments



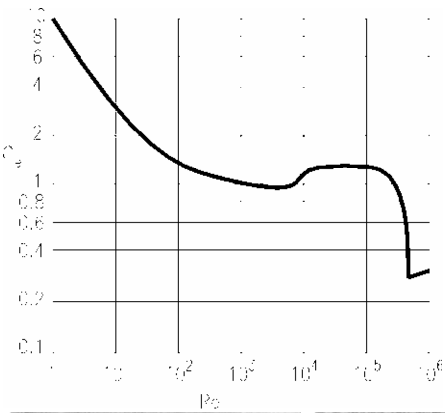
Inviscid fluid: $\left|\underline{p}_2\right| = \left|\underline{p}_1\right|$ Since $p_2 = p_1 = p_\infty$, $v_2 = v_1$, $\Rightarrow$ $\left|\underline{I}_2\right| = \left|\underline{I}_1\right|$. $-I_1 + I_2 = P_1 - P_2 - R_x$, $\Rightarrow$ $R_x = 0$

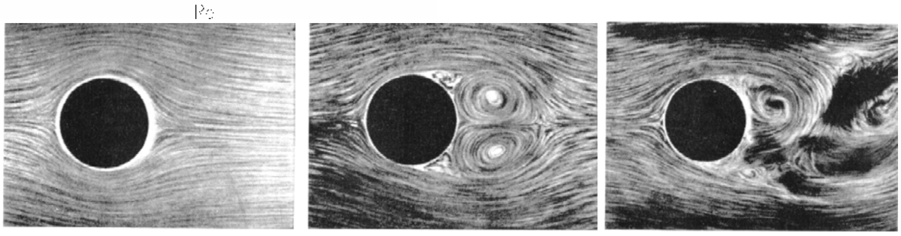
<u>Aerodynamic force acting on cylinder</u>

$f\left(F_d, v_\infty, \rho, \mu, d, \ell\right) = 0$ Dimensionless parameters: $\Pi_1 = c_d = \dfrac{F_d}{\frac{\rho}{2} v_\infty^2 \ell d}$ drag coefficient,

$\Pi_2 = Re = \dfrac{v_\infty d}{\nu}$ Reynolds number, $\Pi_3 = \dfrac{\ell}{d}$ relative length

$\Pi_3 = \dfrac{\ell}{d} = \infty$ 2D flows $\Rightarrow$ $\Pi_1 = f\left(\Pi_2\right)$, $c_d = f\left(Re\right)$

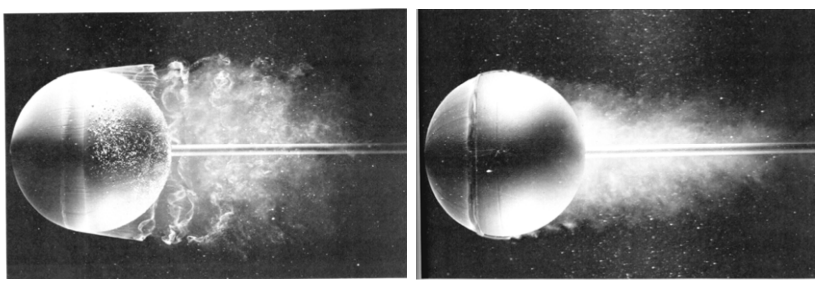




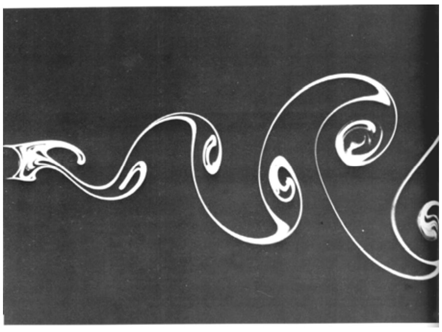
If Re is small, $F_d \sim \mu v_\infty$,

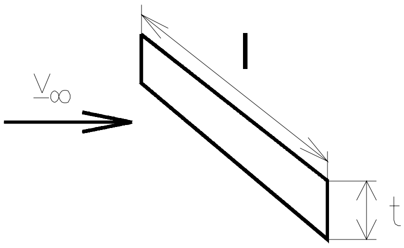
In case of larger Re $F_d \sim v_\infty^2$

Effect of laminar-turbulent BL transition.



Karman vortex street:



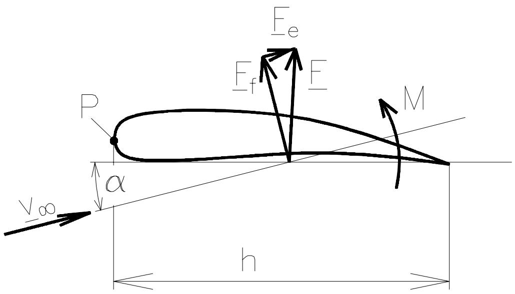


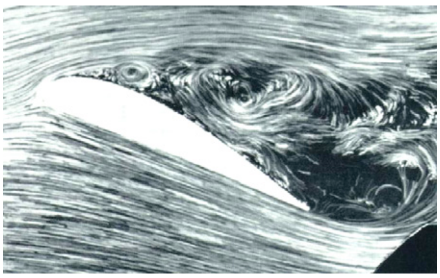
2D flow, $\frac{\ell}{t} \Rightarrow \infty \;\; c_d = 2$. $F_d = (\overline{p}_f - \overline{p}_b)\ell t \Rightarrow c_d = \frac{\overline{p}_f - p_\infty}{\frac{\rho}{2}v_\infty^2} - \frac{\overline{p}_b - p_\infty}{\frac{\rho}{2}v_\infty^2} = \overline{c}_{pf} - \overline{c}_{pb}$

$c_{p\,max} = 1 \;\; \overline{c}_{pf} \cong 0.7 \Rightarrow \overline{c}_{pb} \cong -1.3$

3D effects: $\frac{\ell}{d} = \infty, 10, 1$ c_d = 2, 1.3, 1.1. In case of circular cylinder $c_d = 1.2$, 0.82 és 0.63 (Re=10^5).

<u>Lift and drag on airfoils</u>

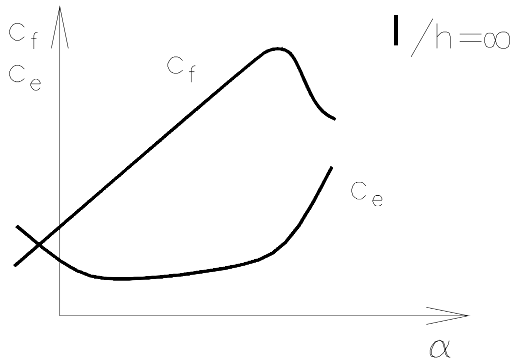




$|\underline{R}| = \rho v_\infty \Gamma \left[\frac{N}{m}\right]$ $c_l = \frac{F_l}{\frac{\rho}{2}v_\infty^2 A}$ lift $c_d = \frac{F_d}{\frac{\rho}{2}v_\infty^2 A}$ drag

$c_{Mp} = \frac{M_p}{\frac{\rho}{2}v_\infty^2 Ah}$ pitching moment coefficient

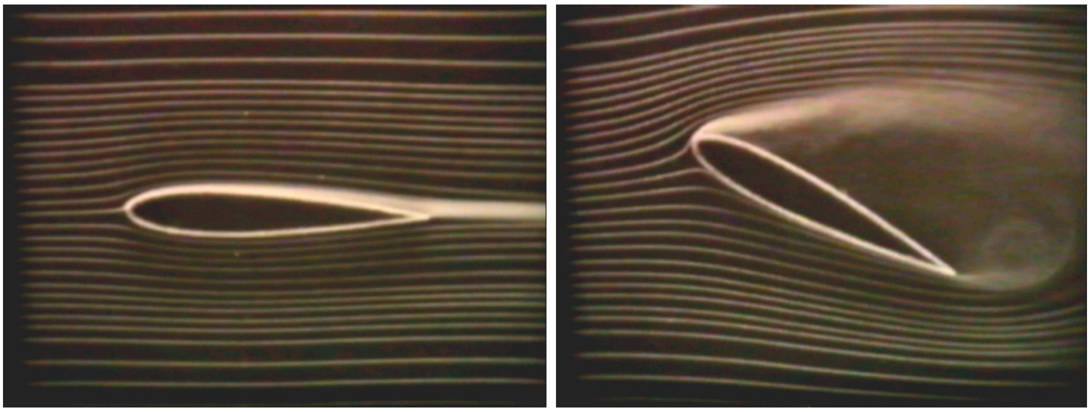
Lift and drag coefficients as function of angle of attack



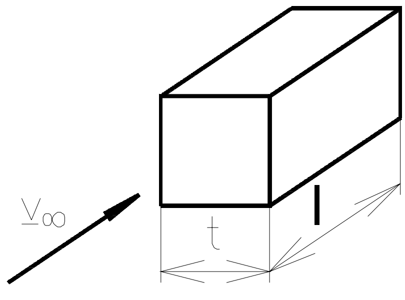
$c_e - c_D$, $c_f - c_L$

$c_{L\,max} \cong 1.2 \sim 1.8$, $dc_L/d\alpha = 2\pi$[rad],

Adverse pressure gradient can cause boundary layer separation



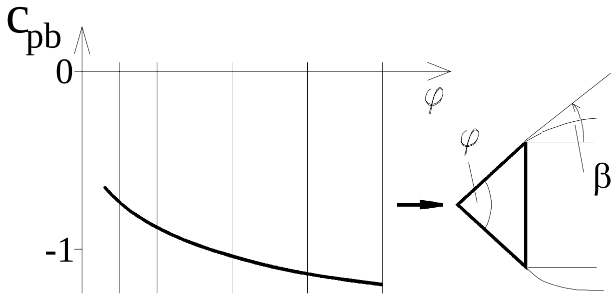
Drag acting on a prismatic bluff body



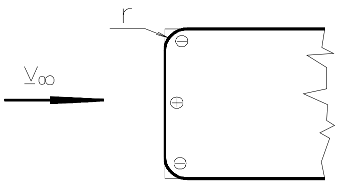
$c_D = \bar{c}_{pf} - \bar{c}_{pb} + 4\dfrac{\ell}{t}\bar{c}'_f$. Drag = forebody drag + base drag + side wall drag

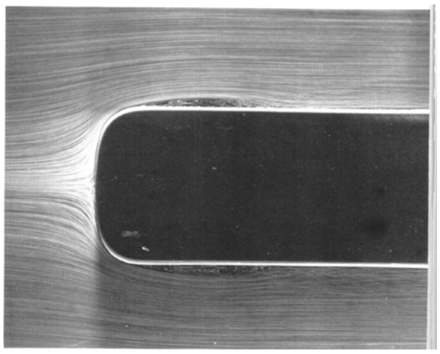
$\ell/t = 0$ and $\ell/t = 5$ c_e = 1.1 and 0.8.

Base pressure coefficient as function of angle between the undisturbed flow and the shear layer connecting to BL separation line.



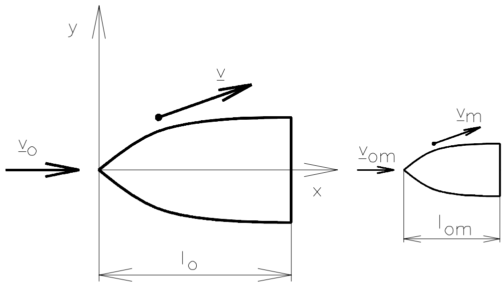
Reduction of forebody drag by rounding up of leading edges.





$\ell/t = 5 \quad c_D = 0.8 \Rightarrow 0.2$

29. Similarity of flows



Characteristic velocities, lengths, times of full scale prototype and model: v_0 and v_{0m}, l_0 and l_{0m}, $t_0 = \frac{\ell_0}{v_0}$ and $t_{0m} = \frac{\ell_{0m}}{v_{0m}}$.

$$\frac{\underline{v}}{v_0} = f\left(\frac{x}{\ell_0}, \frac{y}{\ell_0}, \frac{z}{\ell_0}, \frac{t}{t_0}\right) \text{ and } \frac{p}{\rho v_0^2} = F\left(\frac{x}{\ell_0}, \frac{y}{\ell_0}, \frac{z}{\ell_0}, \frac{t}{t_0}\right).$$

Conditions of similarity:

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \qquad \text{multiplied by } \frac{\ell_0}{v_0^2},$$

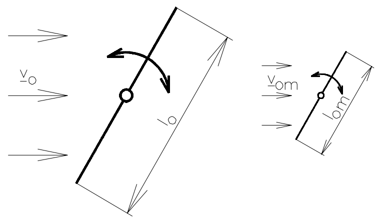
$$\frac{\partial\left(\frac{v_x}{v_0}\right)}{\partial\left(\frac{t}{\ell_0/v_0}\right)} + \frac{v_x}{v_0}\frac{\partial\left(\frac{v_x}{v_0}\right)}{\partial\left(\frac{x}{\ell_0}\right)} + .. = \frac{g_x\ell_0}{v_0^2} - \frac{\partial\left(\frac{p-p_0}{\rho v_0^2}\right)}{\partial\left(\frac{x}{\ell_0}\right)} + \frac{\nu}{v_0\ell_0}\left(\frac{\partial^2\left(\frac{v_x}{v_0}\right)}{\partial\left(\frac{x}{\ell_0}\right)^2} + ..\right) \qquad \text{dimensionless NS equation}$$

Two flows are similar if

a) they are described by the same dimensionless NS equation, i.e.

$\frac{g_x \ell_0}{v_0^2} = \frac{g_{xm} \ell_{0m}}{v_{0m}^2} \Rightarrow Fr = \frac{v_0}{\sqrt{g\ell_0}}$ Froude number, $\frac{\nu}{v_0 \ell_0} = \frac{\nu_m}{v_{0m} \ell_{0m}} \Rightarrow Re = \frac{v_0 \ell_0}{\nu}$ Reynolds number, $Re_m = Re \quad Fr_m = Fr$

b) the initial and boundary conditions are the same in dimensionless form (e.g. geometric similarity of prototype and model).



$t_0 = l_0/v_0,\ \frac{t_{pm}}{t_{0m}} = \frac{t_p}{t_0}$ i.e. $\frac{t_{pm} v_{0m}}{\ell_{0m}} = \frac{t_p v_0}{\ell_0}$.

$t_p = \frac{1}{f}$, f[1/s] frequency $Str = \frac{f\ell_0}{v_0}$ Strouhal number

<u>Dimensionless parameters as ratios of forces acting on unit mass</u>

inertial force: $F_T \sim \frac{v_0^2}{\ell_0}$

field of force: $F_G \sim g$

pressure force $F_P \sim \frac{(p-p_0)\ell_0^2}{\rho\ell_0^3} = \frac{(p-p_0)}{\rho\ell_0}$

viscous force: $F_S \sim \rho\nu \frac{v_0}{\ell_0} \frac{\ell_0^2}{\rho\ell_0^3} = \nu \frac{v_0}{\ell_0^2}$

surface tension force: $F_F \sim \frac{C}{\ell_0} \frac{\ell_0^2}{\rho\ell_0^3} = \frac{C}{\rho\ell_0^2}$

Reynolds number: $Re \sim \frac{F_T}{F_S} \sim \frac{v_0^2/\ell_0}{\nu v_0/\ell_0^2} = \frac{v_0 \ell_0}{\nu}$

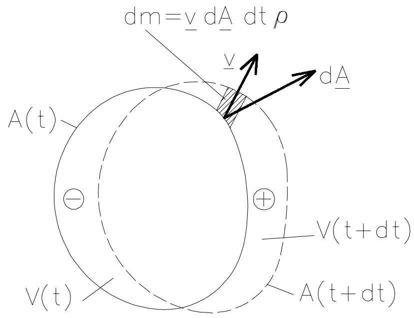
Froude number: $Fr \sim \sqrt{\frac{F_T}{F_G}} = \sqrt{\frac{v_0^2/\ell_0}{g}} = \frac{v_0}{\sqrt{g\ell_0}}$

Euler number: $Eu \sim \frac{F_P}{F_T} \sim \frac{(p-p_0)/\rho/\ell_0}{v_0^2/\ell_0} = \frac{p-p_0}{\rho v_0^2}$

Weber number: $We \sim \frac{F_F}{F_T} \sim \frac{C/\ell_0^2/\rho}{v_0^2/\ell_0} = \frac{C}{\rho v_0^2 \ell_0}$

30. Flow of compressible fluids

Energy equation



inviscid fluid, steady flow, no field of force, no heat transfer

$$\frac{d}{dt}\int_V\left(\frac{v^2}{2}+c_vT\right)\rho dV=-\int_A \underline{v}pd\underline{A}\ ,$$

where $c_v\left[\frac{J}{kgK}\right]$ constant-volume specific heat, $c_vT+\frac{p}{\rho}=h=c_pT$,

h [J/kg] az entalpia, c_p [J/kg/K] constant-pressure specific heat

$\boxed{\frac{v^2}{2}+c_pT=\text{Const.}}$ along streamline

Static, dynamic and total temperature

$$T+\frac{v^2}{2c_p}=T_t=\text{Const.}$$

where T (or T_{st}) [K] static temperature, $T_d=\frac{v^2}{2c_p}$[K] dynamic temperature, T_t[K] total temperature.

Energy equation ⇒ total temperature is constant along a streamline (steady flow of inviscid fluid)

Bernoulli equation for compressible gases

No viscous effects, and no heat transfer ⇒ isentropic flow:

$\frac{p}{\rho^\kappa}=\text{const.}=\frac{p_1}{\rho_1^\kappa}$,$\kappa=\frac{c_p}{c_v}$ ratio of specific heats (isentropic exponent)

Bernoulli equation: $\frac{v_2^2-v_1^2}{2}=-\int_{p_1}^{p_2}\frac{dp}{\rho(p)}\ \Rightarrow\ v_2^2=v_1^2+\frac{2\kappa}{\kappa-1}\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}\right].$

Velocity along a streamline: $\boxed{v_2=\sqrt{v_1^2+\frac{2\kappa}{\kappa-1}RT_1\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}\right]}}$ *

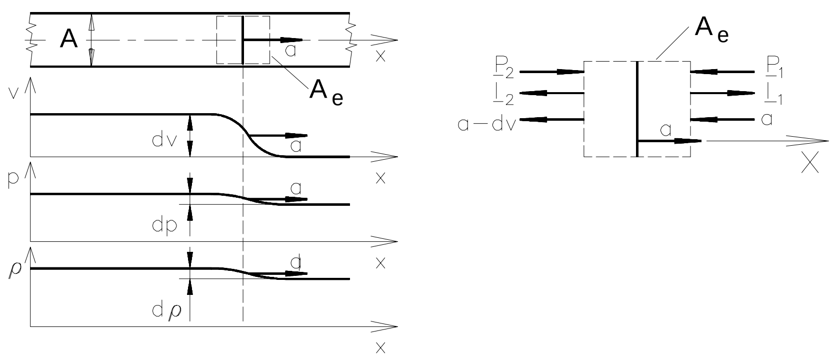
$$\frac{p_1}{\rho_1^\kappa} = \text{const.} = \frac{p_2}{\rho_2^\kappa} \text{ and } \rho_1 = \frac{p_1}{RT_1} \Rightarrow \quad \frac{p_2}{p_1} = \frac{\rho_2^\kappa}{\rho_1^\kappa} = \frac{p_2^\kappa T_1^\kappa R^\kappa}{R^\kappa T_2^\kappa p_1^\kappa} \Rightarrow \boxed{\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}}$$

$\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\kappa}}}$, $\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\kappa-1}}}$. Inserting $\boxed{\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}}$ in * expression we obtain:

$v_2^2 = v_1^2 + \frac{2\kappa}{\kappa-1} RT_1\left[1 - \frac{T_2}{T_1}\right]$. $\frac{2\kappa R}{\kappa-1} = 2c_p, \kappa = \frac{c_p}{c_v}$ $\Rightarrow$ $v_2^2 = v_1^2 + 2c_p(T_1 - T_2)$ the energy equation $c_p T_1 + \frac{v_1^2}{2} = c_p T_2 + \frac{v_2^2}{2}$.

Discharge from a reservoir: $v = \sqrt{\frac{2\kappa}{\kappa-1} RT_t\left[1 - \left(\frac{p_e}{p_t}\right)^{\frac{\kappa-1}{\kappa}}\right]}$

Speed of sound



Integral momentum equation $\rho a^2 A - (\rho + d\rho)(a - dv)^2 A = (p + dp)A - pA$

$2a\rho dv - a^2 d\rho = dp$ Continuity: $(a - dv)(\rho + d\rho) = a\rho \Rightarrow \rho dv = a d\rho$.

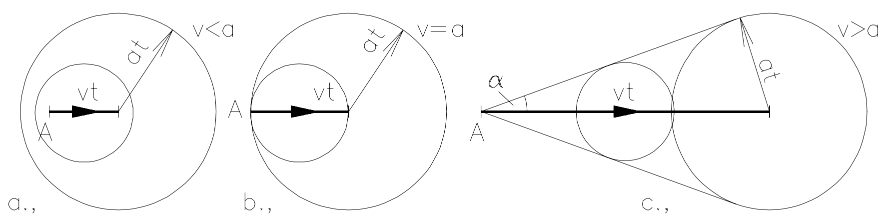
speed of sound: $\boxed{a = \sqrt{\frac{dp}{d\rho}}}$ In case of isentropic process $p = \frac{p_0}{\rho_0^\kappa}\rho^\kappa, \frac{dp}{d\rho} = \frac{p_0}{\rho_0^\kappa}\kappa\rho^{\kappa-1}$

speed of sound: $\sqrt{\frac{dp}{d\rho}} = \boxed{a = \sqrt{\kappa RT}}$

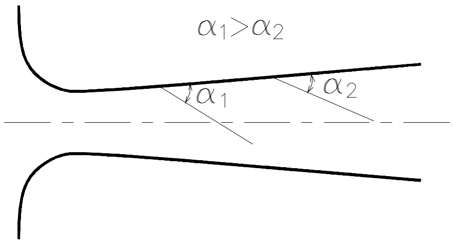
Additional conditions for similarity of flow of compressible fluids:

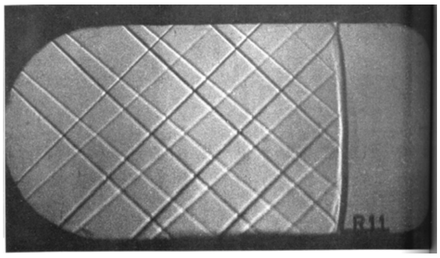
κ = identical and Mach number $Ma = \frac{v_0}{a_0}$ = identical

Propagation of pressure waves



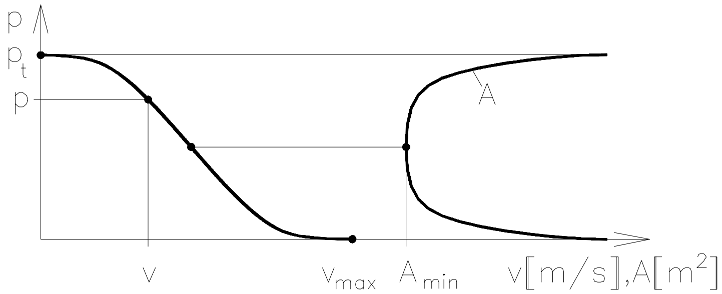
Mach cone $\sin\alpha = \frac{at}{vt} = \frac{a}{v} = \frac{1}{Ma}$





Release of a gas from a reservoir

$$v = \sqrt{\frac{2\kappa}{\kappa - 1} R T_t \left[1 - \left(\frac{p}{p_t} \right)^{\frac{\kappa-1}{\kappa}} \right]}.$$



$v_{max} = \sqrt{\frac{2\kappa}{\kappa - 1} R T_t}$ tangent of the curve $v\frac{\partial v}{\partial e} = -\frac{1}{\rho}\frac{\partial p}{\partial e} + g_e \Rightarrow \frac{dp}{dv} = -\rho v$.

max $-\frac{dp}{dv} \Rightarrow \frac{d(\rho v)}{dv} = v\frac{d\rho}{dv} + \rho = 0 \Rightarrow \rho\left[1 - \frac{v^2}{dp/d\rho}\right] = \rho\left[1 - \frac{v^2}{a^2}\right] = 0$

in point of inflexion v = a, i.e. $Ma = 1$

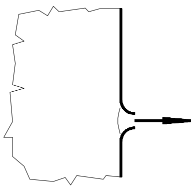
Since $q_m = \rho v A = -\frac{dp}{dv} A = \text{const.}$ at $q_m = \rho v A = -\frac{dp}{dv} A = \text{const.}$ (at $v = 0$ and $p = 0$) $A \to \infty$.

At $-\frac{dp}{dv}$ = max. the cross section A is the smallest (throat, critical area): Laval nozzle

$$T_t = T^* + \frac{a^{*2}}{2c_p} = T^* + \frac{\kappa R T^*}{2c_p} = T^*\left[1 + \frac{c_p / c_v (c_p - c_v)}{2c_p}\right] = \frac{\kappa + 1}{2} T^*$$

$$\frac{T^*}{T_t} = \frac{2}{\kappa + 1}\ (= 0.833),\ \frac{p^*}{p_t} = \left(\frac{T^*}{T_t}\right)^{\frac{\kappa}{\kappa-1}} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}}\ (= 0.53)\ \frac{\rho^*}{\rho_t} = \left(\frac{T^*}{T_t}\right)^{\frac{1}{\kappa-1}} = \left(\frac{2}{\kappa+1}\right)^{\frac{1}{\kappa-1}}\ (= 0.63).$$

Simple discharge nozzle



$p_e < p_t$ if $p_e / p_t > 0.95$ $\rho \cong \text{const.} \Rightarrow v = \sqrt{\frac{2}{\rho}(p_t - p_e)}$

if $p_e/p_t < 0.95$ $\rho \neq$ const. $\Rightarrow$ $v = \sqrt{\frac{2\kappa}{\kappa-1} RT_t \left[1 - \left(\frac{p}{p_t}\right)^{\frac{\kappa-1}{\kappa}}\right]}$

if $p_e/p_t = 0.53$ (in case of $\kappa = 1.4$), $v^* = a^* = \sqrt{\kappa R T^*}$ where $T^* = 0.833 T_t$.
if $p_e/p_t < 0.53$ the process is same as in case of $p_e/p_t = 0.53$. ($p^* = 0.53 p_t$.)

Flow in Laval nozzle

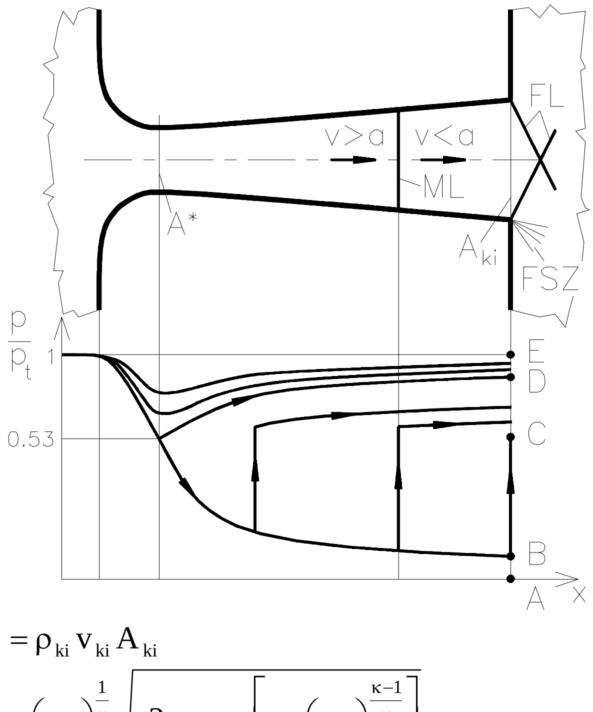
$$v\frac{\partial v}{\partial e} = -\frac{1}{\rho}\frac{\partial p}{\partial e} = -\frac{1}{\rho}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial e} = -\frac{1}{\rho}a^2\frac{\partial \rho}{\partial e}.$$

$$d\rho v A + \rho dv A + \rho v dA = 0 \Rightarrow \frac{d\rho}{\rho} + \frac{dv}{v} + \frac{dA}{A} = 0 \Rightarrow vdv = -a^2\frac{d\rho}{\rho}.$$

$$vdv = -a^2\frac{d\rho}{\rho} = a^2\left(\frac{dv}{v} + \frac{dA}{A}\right).$$

$$\frac{v^2}{a^2}\frac{dv}{v} = \frac{dv}{v} + \frac{dA}{A} \quad \Rightarrow \boxed{\left(Ma^2 - 1\right)\frac{dv}{v} = \frac{dA}{A}}$$

α/ if $Ma < 1$, in case of $dv/v > 0 \Rightarrow dA/A < 0$. At $dv/v < 0$) $dA/A > 0$
β/ If $Ma > 1$, in case of $dv/v > 0 \Rightarrow dA/A > 0$
γ/ if $dv/v > 0$, and $dA/A \Rightarrow Ma = 1$.
δ/ if $dA/A = 0$ and $Ma \neq 1$ the velocity is extreme.



continuity $q_m = \rho^* v^* A^* = \rho_{ki} v_{ki} A_{ki}$

$$0.63\rho_t\sqrt{\kappa R 0.83 T_t}\,A^* = A_{ki}\rho_t\left(\frac{p_e}{p_t}\right)^{\frac{1}{\kappa}}\sqrt{\frac{2\kappa}{\kappa-1}RT_t\left[1-\left(\frac{p_e}{p_t}\right)^{\frac{\kappa-1}{\kappa}}\right]}$$

Two isentropic solutions: points B and D.