SOCRATES Teaching Staff Mobility Program
2000 - 2001

Technical University of Budapest
1782
Department of Fluid Mechanics

DMA-URLS

Finite Element Method
for Turbomachinery Flows

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***BUDAPEST University of Technology and Economics* - 28 November 2000**

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- FLUID MODELLING
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WHY FEM IN TURBOMACHINERY *CFD* (1)

- FEW FEM APPLICATIONS CONCERN WITH TURBOMACHINERY CFD

- GEOMETRIC FLEXIBILITY ON COMPLEX DOMAIN
  - ISO-PARAMETRIC MAPPING
  - CARTESIAN FIELD DESCRIPTION
  - IMPLICIT UNSTRUCTURED MESHING (EVEN FREE-MESH ALGORITHM)

    NO STRUCTURED MULTI-BLOCK OR EMBEDDING TECHNIQUES ARE REQUIRED

- BOUNDARY CONDITIONS ACCURACY
  - TRUE DIRICHLET CONDITIONS
  - NATURAL OUTFLOW CONDITIONS
  - PHYSICAL PERIODIC CONDITIONS

WHY FEM IN TURBOMACHINERY *CFD* (2)

- STABILITY AND ACCURACY, CONSISTENT PETROV-GALERKIN (PG) FORMULATIONS
  - ♦ SU/PG (Brooks and Hughes, 1982), ADVECTIVE-DIFFUSIVE LIMIT INSTABILITIES

    NON-OSCILLATORY COUNTERPART OF GALERKIN METHODS
  - ♦ PS/PG (Tezduyar, 1992), PURE DIFFUSIVE LIMIT INSTABILITIES

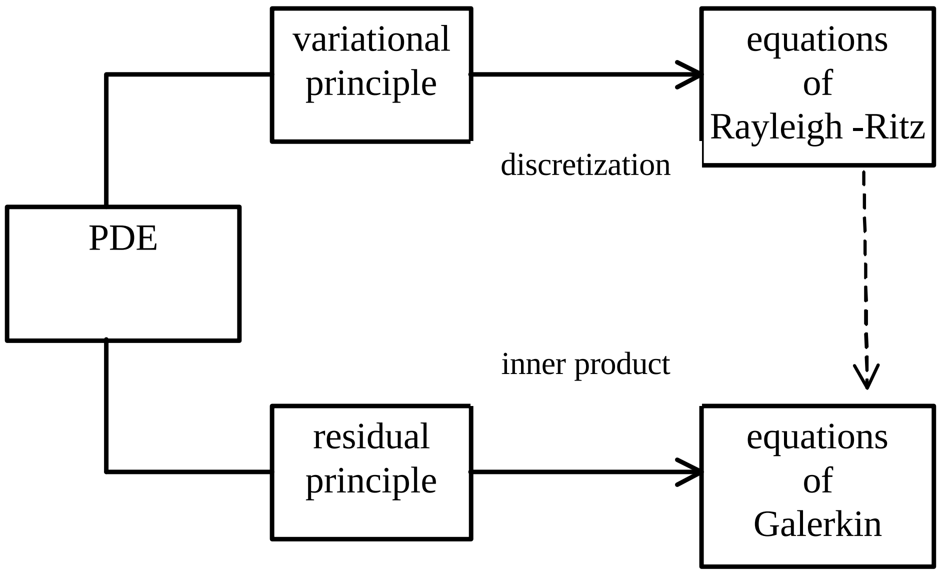
    CIRCUMVENTING THE BABUSKA-BREZZI CONDITION FOR Q1-Q1 ELEMENTS
  - ♦ XENIOS PG FORMULATION (Corsini, 1996), (Borello, Corsini and Rispoli, 1997) ON Q2-Q1 ELEMENTS

    IMPROVED MATRIX CONDITION CHARACTERISTIC, RAPID CONVERGENCE

    ACCURATE ELEMENT-WISE MODELLING OF HIGHER ORDER TERMS OF ANISOTROPIC EVM

- GLOBAL PHYSICAL LAWS SATISFACTION

THE INTEGRAL APPROXIMATION METHODS



- VARIATIONAL METHODS → *ENERGY EQUILIBRIUM* CONDITION
- RESIDUAL METHODS → *ORTHOGONAL PROJECTION* CONDITION

THE RESIDUAL METHODS

Given a Generic Differential Operator

$$Lu - f = 0 \qquad W \in R^{nsd}$$

Dirichelet conditions $\quad u = g \quad$ on G_g

Neumann conditions $\quad u_{,n} = h \quad$ on G_h $\Longrightarrow$

Define two collections of functions

S^h, *trial or candidate* ® $\tilde{u}$

W^h, *weight or variation* ® $\tilde{w}$

APPROXIMATE the solution with *TRIAL* $\qquad L\tilde{u} - f = e$

BUILD an inner product as an ORTHOGONALITY conditions $\qquad (e, \tilde{w}) = \int_W e \cdot \tilde{w} \; \mathrm{d}W = 0$

THE *WEAK* FORM OF NAVIER-STOKES PROBLEM (1)

- WEAK GLOBAL APPROACH *(Hughes, 1987)*
- TRIALS and WEIGHTS generalized properties of CONTINUITY and INTEGRABILITY over $\overline{W} = W \cup G \subset R^{nsd}$
- APPROXIMATE collections

$$S^h = \{\tilde{u} \mid \tilde{u} \in H^{1h}, \tilde{u} = g \rightarrow G_g\}$$

$$W^h = \{\tilde{w} \mid \tilde{w} \in H^{1h}, \tilde{w} = 0 \rightarrow G_g\}$$

with $H^{1h}(W)$ SOBOLEV SPACE

FUNCTION SPACES

- Square integrable function space

$L^2(W)$, space functions square integrable over W: $(p,q)=\int_W pq\,dW<\infty$ $\quad \|q\|_0=(q,q)^{1/2}$

- Sobolev space

$$H^k(W)=\left\{q\in L^2(W): D^s q\in L^2(W), s=1,...k\right\} \qquad \|q\|_k=(\|q\|_0^2+\sum\|D^s q\|_0^2)^{1/2}$$

$H^0(W)=L^2(W)$, $H^1(W)$ *first derivative square integrable,*

$$H_0^1(W)=\left\{q\in H^1(W): q=0\rightarrow G\right\} \qquad \|q\|_1=(\|q\|_0^2+\sum_i^{nsd}\|\partial q/\partial x_i\|_0^2)^{1/2}$$

$H^{1/2}(W)$ restriction to the boundary G of functions $H^1(W)$ $\quad \|q\|_{1/2,G}=\inf\|v\|_1, v\in H^1(W)$

$v=q$ on G

THE *WEAK* FORM OF NAVIER-STOKES PROBLEM (2)

- Residual formulation of INCOMPRESSIBLE N-S BOUNDARY VALUE PROBLEM

$$\int_W \boldsymbol{w}_{ns}\left(\boldsymbol{u}_j \boldsymbol{u}_{i,j}\right) dW - \int_W \boldsymbol{w}_{ns}(\boldsymbol{s}_{ij})_{,i}\, dW = \int_W \boldsymbol{w}_{ns}\, \boldsymbol{f}_i\, dW$$

$$\int_W w_c\, \boldsymbol{u}_{i,i}\, dW = 0$$

$$\boldsymbol{u}_i = \boldsymbol{g}_i \rightarrow G_g$$

$$s_{ij}\, \boldsymbol{n}_i = \boldsymbol{h}_i \rightarrow G_h$$

second order terms

- LOWERING the order of DIFFUSIVE INTEGRAL (Green-Gauss theorem)

$$\int_W \boldsymbol{w}_{ns}\left(s_{ij}\right)_{,j} dW = -\int_W \boldsymbol{w}_{ns,i}\, s_{ij}\, dW + \int_G \boldsymbol{w}_h\, s_{ij}\, \boldsymbol{n}_j\, dG$$

THE *WEAK* FORM OF NAVIER-STOKES PROBLEM (3)

- WEAK Residual form of INCOMPRESSIBLE N-S BOUNDARY VALUE PROBLEM

$$\int_W \boldsymbol{w}_{ns}\left(\boldsymbol{u}_j \boldsymbol{u}_{i,j}\right) dW + \int_W \boldsymbol{w}_{ns,i}\, \boldsymbol{s}_{ij}\, dW = \int_W \boldsymbol{w}_{ns}\, \boldsymbol{f}_i\, dW + \int_G \boldsymbol{w}_h \boldsymbol{h}_i\, dG$$

$$\int_W w_c\, \boldsymbol{u}_{i,i}\, dW = 0$$

$$\boldsymbol{u}_i = \boldsymbol{g}_i \rightarrow G_g$$

natural BCs

essential BCs

- compact WEAK Residual form

$$\int_W \boldsymbol{w}_{ns}\left(\boldsymbol{u}_j \boldsymbol{u}_{i,j}\right) dW + \int_W \boldsymbol{w}_{ns,i}\, \boldsymbol{s}_{ij}\, dW - \int_W \boldsymbol{w}_{ns}\, \boldsymbol{f}_i\, dW - \int_G \boldsymbol{w}_h \boldsymbol{h}_i\, dG + \int_W w_c\, \boldsymbol{u}_{i,i}\, dW = 0$$

continuity scalar equation acts as an additional constraint

w_c as the Lagrangian multiplier of the incompressibility constraint

FINITE ELEMENT METHOD AND DISCRETIZATION (1)

- How the collection S^h and W^h are composed?

for each $s \in S^h$ on $\overline{W} = W \cup G$

$$s\Big|_{G_g} = v\Big|_{G_g} + g \;\rightarrow\; v\Big|_{G_g} = 0$$

$$\Downarrow$$

- The *same* approximate function *v* could be the BASIS for TRIAL and WEIGHTS

v is called the BASIS or SHAPE function

- The FEM based on such a choice is named the GALERKIN residual method.

FINITE ELEMENT METHOD AND DISCRETIZATION (2)

- Let consider a domain $\overline{W} \in R^{nsd}$ and a set of nodal points $l \subset \overline{W}$
- The APPROXIMATE GLOBAL functions have a linear POLYNOMIAL form

$$w = \sum_{l=1}^{nodes} c_l f_l \rightarrow \textbf{weighting} \text{ functions}$$

$$s = \sum_{l=1}^{nodes} d_l f_l \rightarrow \textbf{trial} \text{ functions}$$

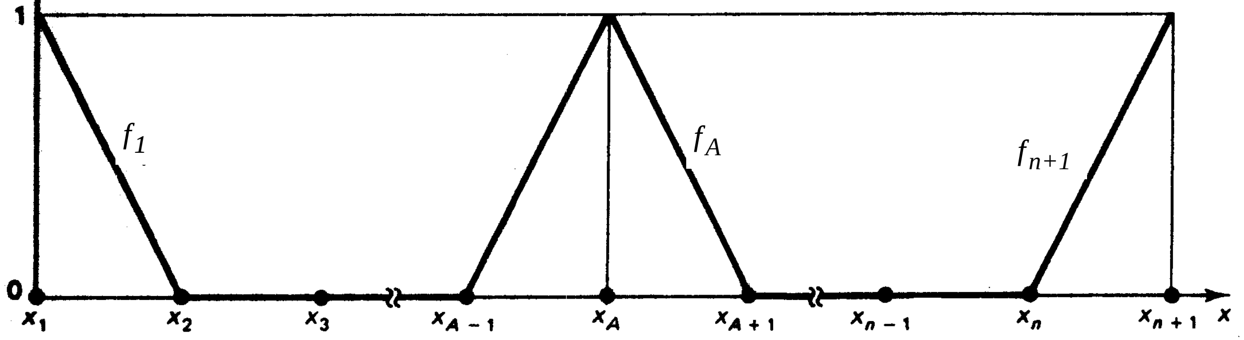
- The interpolating SHAPE function f_l fulfills the nodal properties

$$f_l \big|_l = 1 \rightarrow f_l \big|_m = 0 \quad \forall\, m \subset [\, W \,] \neq l$$

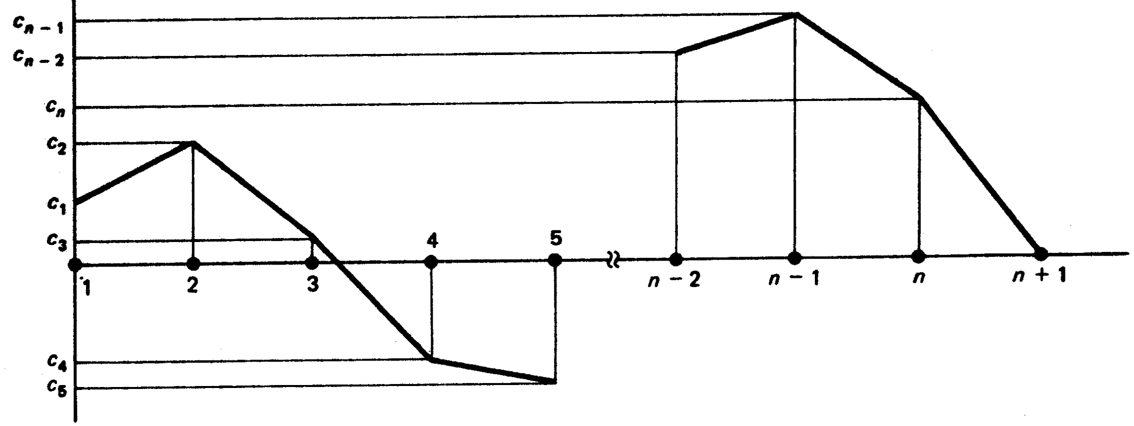
FINITE ELEMENT METHOD AND DISCRETIZATION (3)

- Let consider a 1-D domain $\overline{W} \in R^{nsd}$ and a set of nodal points $l \subset \overline{W}$

nodal linear *shape* functions

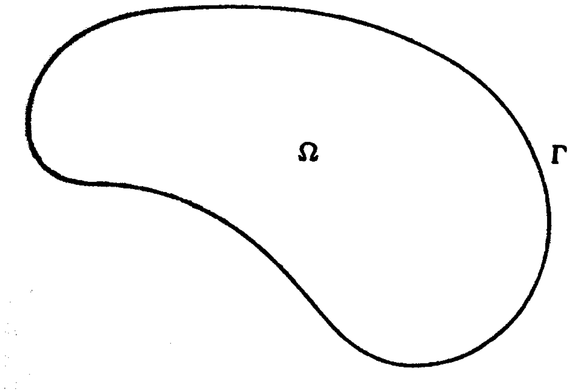


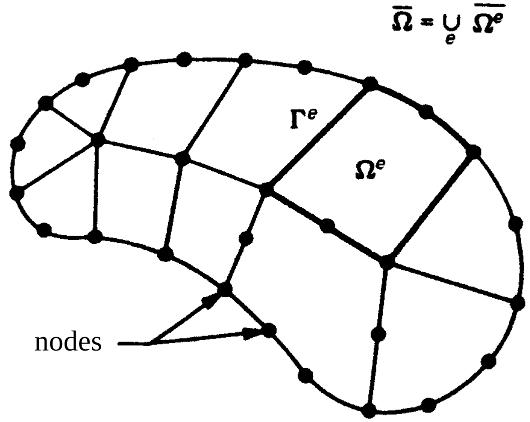
global *interpolated* functions



FINITE ELEMENT METHOD AND DISCRETIZATION (4)

- Let consider the domain subdivision into elements





$$\cup_e \overline{W}_e = W$$

$$\cap_e \overline{W}_e = \varnothing$$

$$G_{int} = \cup_e G_e - G$$

- The GLOBAL weak residual formulation could be then discretized onto the domain

$$\underset{e}{S} \int_{W_e} \boldsymbol{w}_{ns}\left(r\, \boldsymbol{u}_j \boldsymbol{u}_{i,j}\right) dW + \underset{e}{S} \int_{W_e} \boldsymbol{w}_{ns,i}\, s_{ij}\, dW - \underset{e}{S} \int_{G_{int}} \boldsymbol{w}_{ns} [s_{ij} \boldsymbol{n}_i]\, dG + \underset{e}{S} \int_{W_e} w_c \boldsymbol{u}_{i,i}\, dW = \underset{e}{S} \int_{W_e} \boldsymbol{w}_{ns} \boldsymbol{f}_i\, dW + \int_{G_h} \boldsymbol{w}_h \boldsymbol{h}_i\, dG$$

$$\boldsymbol{u_i} = g_i \rightarrow G_g$$

balance of inter-element diffusive fluxes

FINITE ELEMENT METHOD AND DISCRETIZATION (5)

- Let now consider the residual NS problem written for a generic elementary domain Ω_e

$$\int_{\Omega_e} \boldsymbol{w}_{ns}\left(\rho\, \boldsymbol{u}_j \boldsymbol{u}_{i,j}\right) d\Omega + \int_{\Omega_e} \boldsymbol{w}_{ns,i}\, \sigma_{ij}\, d\Omega - \int_{\Gamma_e} \boldsymbol{w}_{ns}\, \sigma_{ij}\, \boldsymbol{n}_i\, d\Gamma - \int_{\Omega_e} \boldsymbol{w}_{ns}\, \boldsymbol{f}_i\, d\Omega + \int_{\Omega_e} w_c\, \boldsymbol{u}_{i,i}\, d\Omega = 0$$

where the weights $\boldsymbol{w}$ and the trial $\boldsymbol{u}$ are *simply* C^0 on each element

- Composing each elementary formulations the residual GLOBAL form could be build as

$$\Sigma_e \int_{\Omega_e} \boldsymbol{w}_{ns}\left(\rho\, \boldsymbol{u}_j \boldsymbol{u}_{i,j}\right) d\Omega + \Sigma_e \int_{\Omega_e} \boldsymbol{w}_{ns,i}\, \sigma_{ij}\, d\Omega - \Sigma_e \int_{\Gamma_e} \boldsymbol{w}_{ns}\, \sigma_{ij}\, \boldsymbol{n}_i\, d\Gamma + \Sigma_e \int_{\Omega_e} w_c\, \boldsymbol{u}_{i,i}\, d\Omega = \Sigma_e \int_{\Omega_e} \boldsymbol{w}_{ns}\, \boldsymbol{f}_i\, d\Omega$$

where

$$\Sigma_e \int_{\Gamma_e} \boldsymbol{w}_{ns}\, \sigma_{ij}\, \boldsymbol{n}_i\, d\Gamma = \Sigma_e \int_{\Gamma_{int}} \boldsymbol{w}_{ns} \left[\sigma_{ij}\, \boldsymbol{n}_i\right] d\Gamma + \int_{\Gamma_h} \boldsymbol{w}_h\, \sigma_{ij}\, \boldsymbol{n}_i\, d\Gamma$$

STEPS TO A FINITE ELEMENT METHOD

- the approximate representation of CONTINUOUS COMPUTATIONAL DOMAIN W, as the composition of small sub-domains called **elements** W_e

- the definition of APPROXIMATE SOLUTION, obtained by the interpolation of the unknown values in a **finite number of nodal points** defined in each element using a collection of **basis functions**

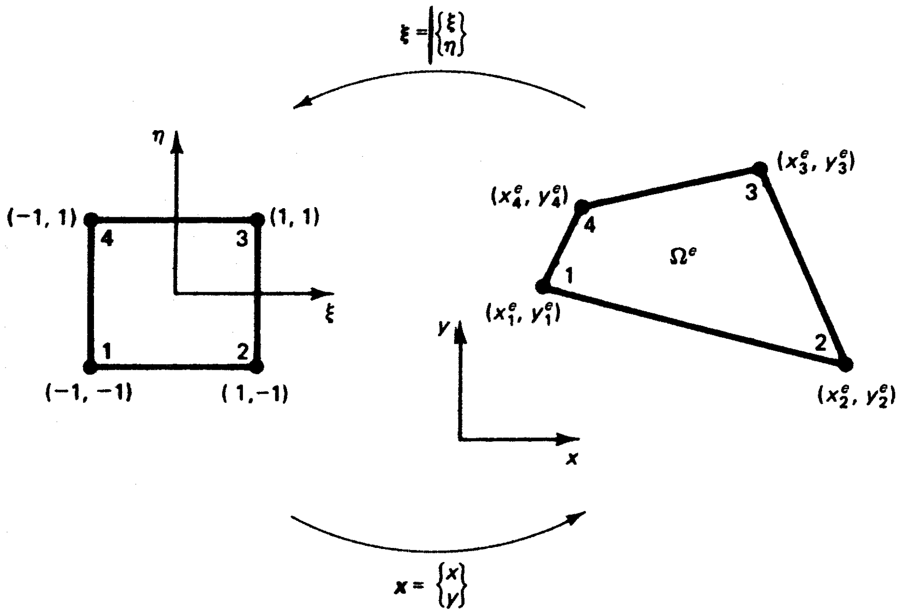
- the definition of an INTEGRAL EQUATION for each unknown variable, by use of a **residual principle**

FINITE ELEMENT INTERPOLATION FUNCTIONS (1)

- Modeling complex geometries introduces DISTORTIONS of the elementary sub-domains

  recall the FEM capability of working on **unstructured&distorted** grids

- from a CODING VIEWPOINT it is mandatory to introduce a space correlation between R^{nsd} and LOGIC FRAME OF REFERENCE (x, h, z)



mapping or *coordinate transformation operator*

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = f \begin{Bmatrix} x \\ h \\ V \end{Bmatrix}$$

FINITE ELEMENT INTERPOLATION FUNCTIONS (2)

- ELEMENT-BY-ELEMENT the domain geometry is constructed using appropriate transformation *polynomial* operators

$$x(x,h,V) = \sum_{i=1}^{nel} N_i(x,h,V)x_i$$

$$y(x,h,V) = \sum_{i=1}^{nel} N_i(x,h,V)y_i$$

$$z(x,h,V) = \sum_{i=1}^{nel} N_i(x,h,V)z_i$$

element shape functions

- The SHAPE functions N_i could be defined using a *direct formulation* imposing

  $N_i(i) = 1$, and $N_i(j) = 0$ for *j* ¹ *i*

  the continuity of element functions along the element boundaries

- ISOPARAMETRIC elements use identical shape functions for geometry&DOF

FINITE ELEMENT INTERPOLATION FUNCTIONS (3)

- Shape functions DERIVATIVES (first order) *simply* defined in the LOGIC reference

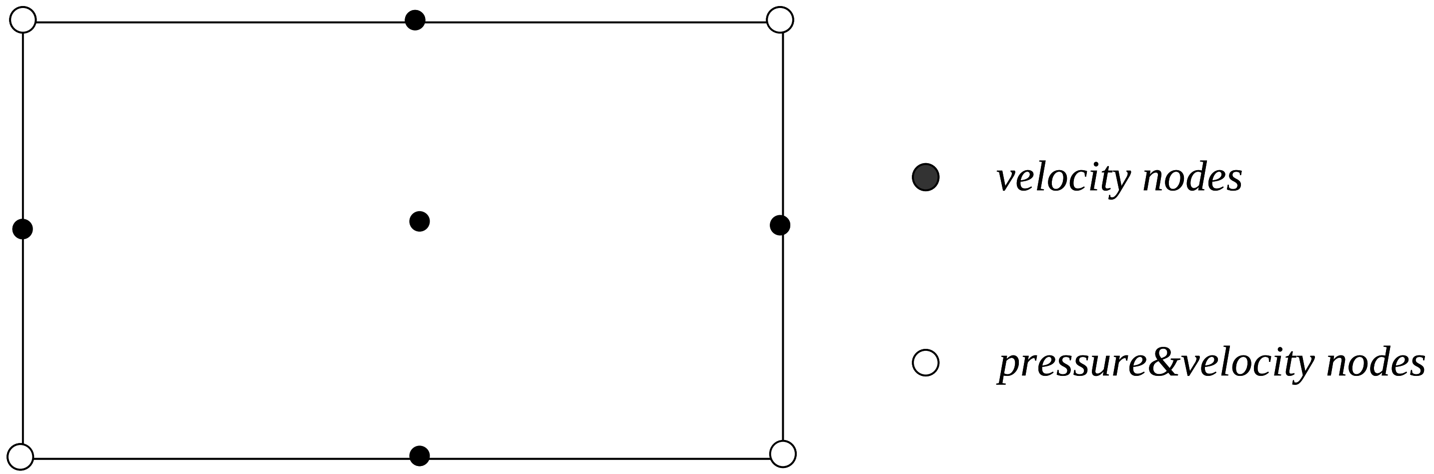
$$\begin{Bmatrix} \frac{\partial \Phi}{\partial \xi} \\ \frac{\partial \Phi}{\partial \eta} \\ \frac{\partial \Phi}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{Bmatrix} = J \begin{Bmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{Bmatrix} \qquad \Rightarrow \qquad \begin{Bmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial \Phi}{\partial \xi} \\ \frac{\partial \Phi}{\partial \eta} \\ \frac{\partial \Phi}{\partial \zeta} \end{Bmatrix}$$

- Due to the intrinsic simplicity of LOGIC geometry&DOF definition the computation of integral terms is performed as

$$\int_V K dV = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} K \left[det\, J \right] d\xi \, d\eta \, d\zeta$$

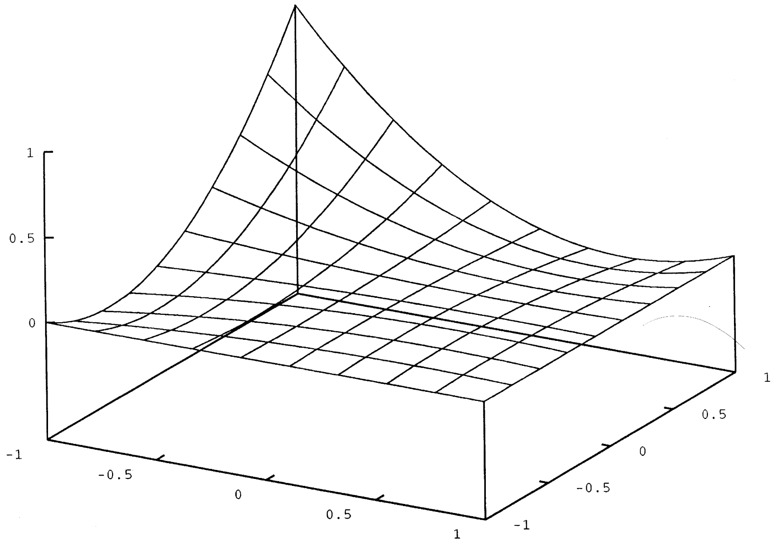
INTERPOLATION SPACES in XENIOS (1)

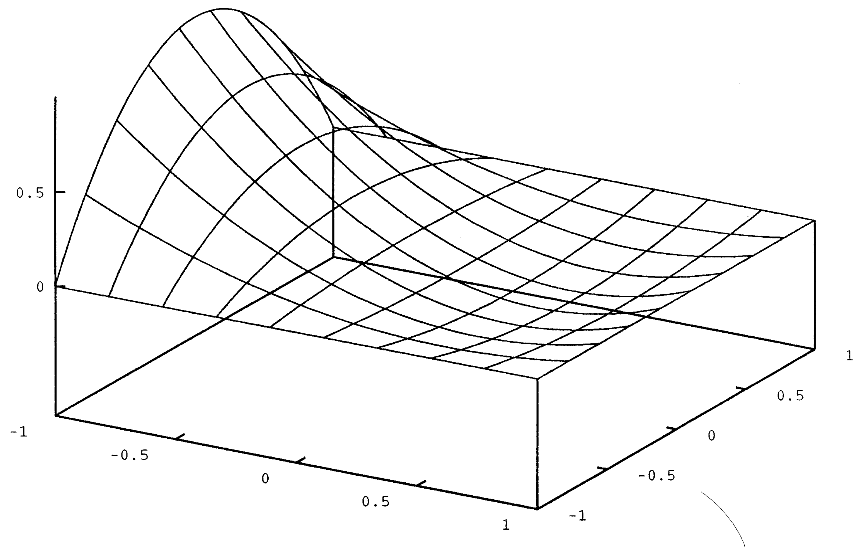
- MIXED finite element approximations spaces

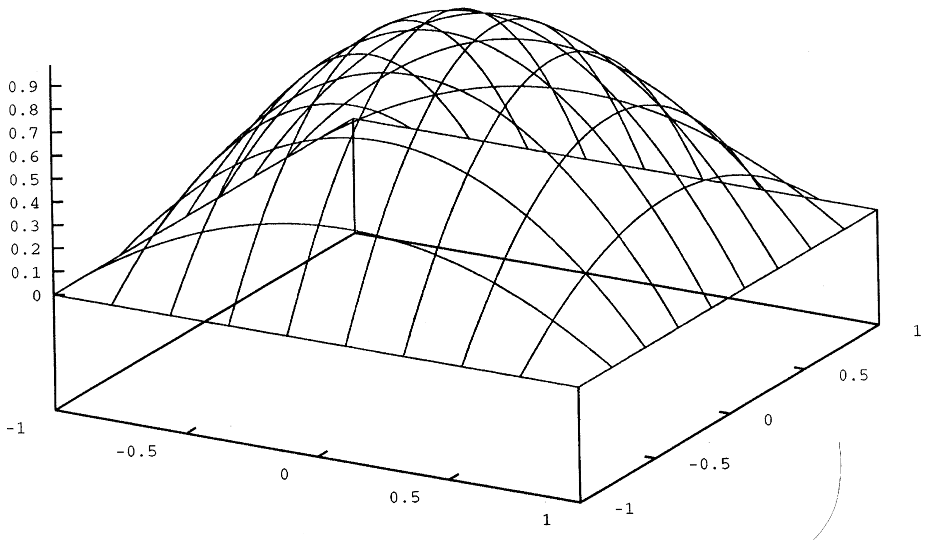


- quadratic approx for **velocity** DOFs, linear approx for **pressure** DOF

INTERPOLATION SPACES in XENIOS (2)

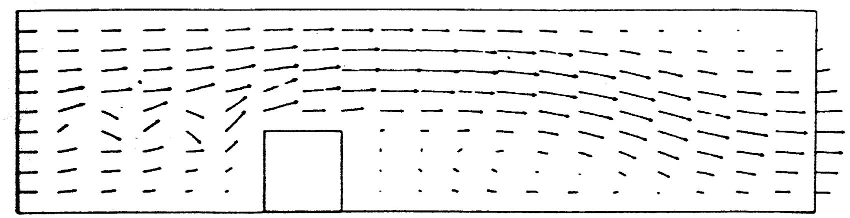




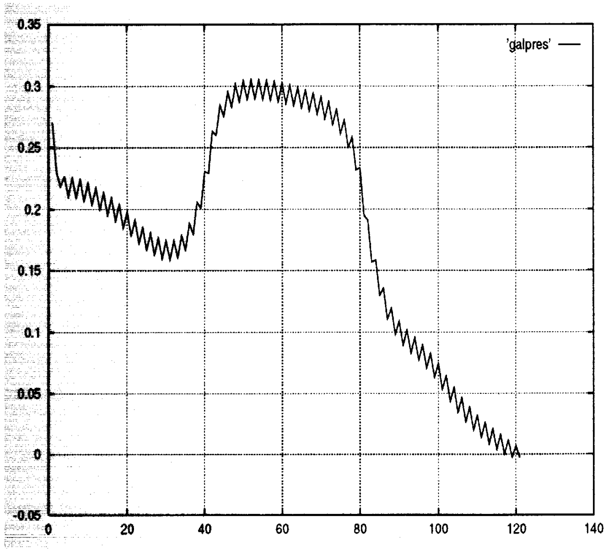


REASONS TO STABILIZE A FEM FORMULATION (1)

- purely *advective flow limit*, the localization of advective transport of variable along the streamlines



- purely *diffusive flow limit* (*Stokes flow*), pressure instability related to the incompressibility constraint



ADVECTIVE LIMIT INSTABILITY ORIGIN

- is not possible to simulate strongly asymmetric terms by use of SYMMETRIC operators

Galerkin shape functions or equivalently centered FD stencils

- for 1-D the **stability condition** for numerical approx. of convective term is

$$\frac{\partial}{\partial \varphi}\left(\boldsymbol{u}\cdot\frac{\partial \varphi}{\partial x}\right)<0 \quad \Rightarrow \quad \boldsymbol{u}\cdot\frac{\partial \varphi}{\partial x} \approx \boldsymbol{u}\frac{\varphi_{i+1}-\varphi_{i-1}}{2\Delta x}$$

neutral stability to φ_i

- classical approach to recover the stability models the convective term *via UPWIND* FD

$$\boldsymbol{u}\cdot\frac{\partial \varphi}{\partial x} \approx \boldsymbol{u}\frac{\varphi_{i}-\varphi_{i-1}}{\Delta x} = \boldsymbol{u}\frac{\varphi_{i+1}-\varphi_{i-1}}{2\Delta x} + \frac{\boldsymbol{u}\Delta x}{2}\frac{-\varphi_{i+1}+2\varphi_{i}-\varphi_{i-1}}{\Delta x^{2}}$$

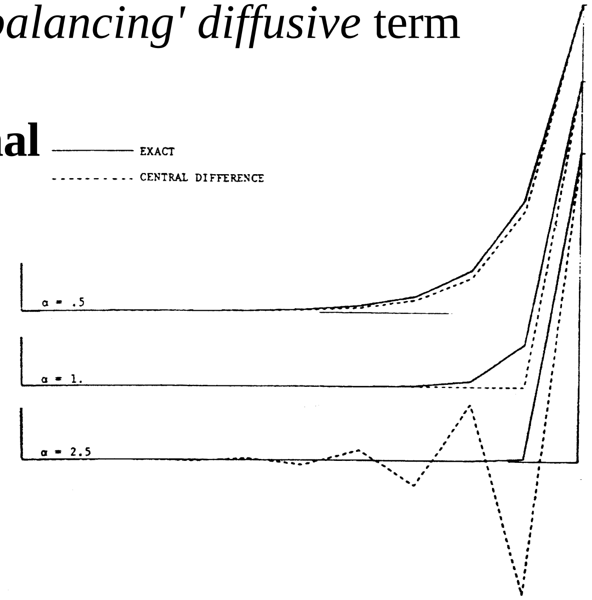
centered first and second order terms

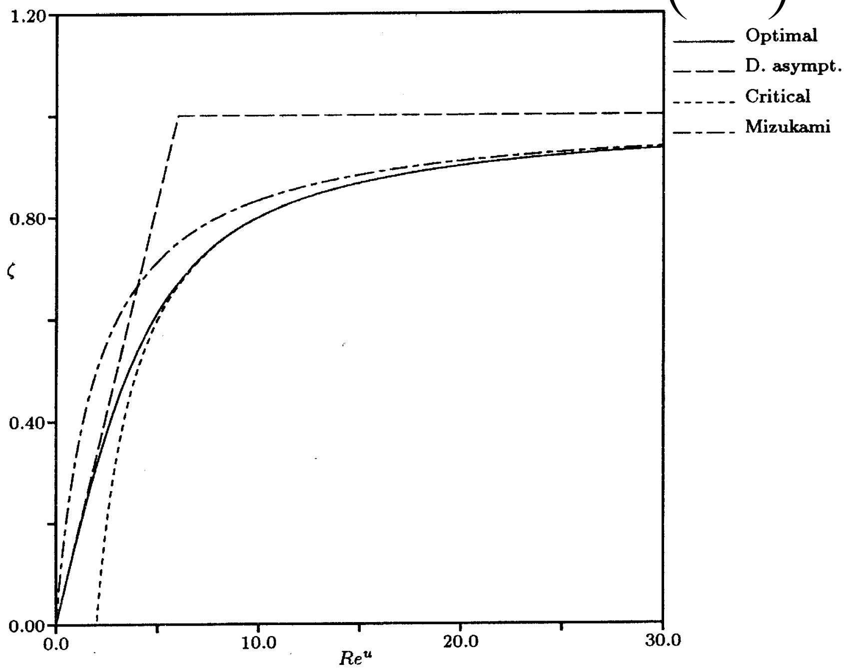
NON CONSISTENT STABILIZED FEM FORMULATION

- Galerkin FEM could be stabilized introducing an *artificial 'balancing' diffusive* term

  in 1-D the diffusive like *scalar upwind* stabilization is **optimal**

  optimal *scalar artificial diffusivity* $\tilde{k} = \left(\frac{|u|h}{2} \right) z$





Comparison of different $\zeta(Re^u)$ functions.

$z = coth(a) - 1/a$, *magic* function

$a = \frac{|u|h}{2k}$, grid or local Peclet number

NON CONSISTENT *STREAMLINE UPWIND* FEM FORMULATION

- in multi-dimensionale cases a *STREAMLINE upwind* correction in mandatory

  the **balancing operator** depends on a *tensorial artificial diffusivity* $\tilde{k}_{ij} = \tilde{k} \cdot \bar{\boldsymbol{u}}_i \bar{\boldsymbol{u}}_j$

- **residual 'interpretation'** of *streamline upwind* term

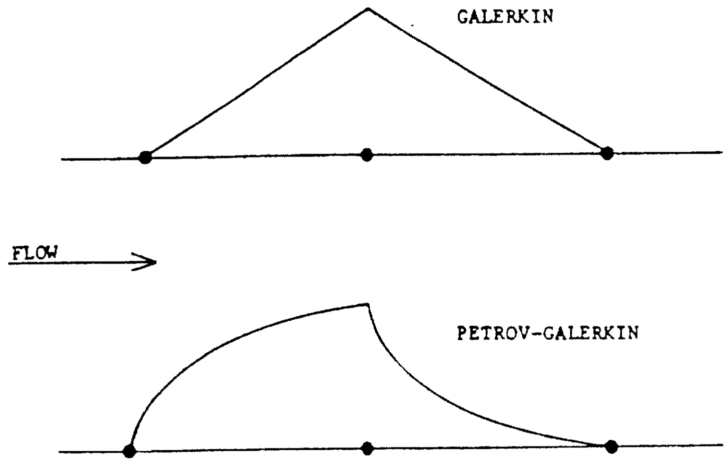
$$\int_W \boldsymbol{w}_{ns,i} k_{ij}\, s_{ij}\, dW + \int_W \boldsymbol{w}_{ns,i} \tilde{k}_{ij}\, u_{i,j}\, dW \qquad \int_W \boldsymbol{w}_{ns,i} \tilde{k}\, \bar{u}_i \bar{u}_j u_{i,j}\, dW$$

$$\int_W \boldsymbol{w}_{ns,i} \frac{\bar{u}_i}{\|u\|} \tilde{k}\, u_j u_{i,j}\, dW$$

perturbed weight for the convective integral

CONSISTENT *STREAMLINE UPWIND-PETROV GALERKIN* FEM FORMULATION (1)

- the consistency of the stabilization method is recovered by perturbating the Galerkin weights in such a way that a *Petrov-Galerkin residual formulation* is achieved



$$\boldsymbol{w_{ns}}' = \boldsymbol{w_{ns}} + \boldsymbol{p_{ns}}$$

CONSISTENT *STREAMLINE UPWIND-PETROV GALERKIN* FEM FORMULATION (2)

- SUPG formulation of Navier-Stokes problem

$$\Sigma_e \int_{\Omega_e} [\boldsymbol{w}_{ns}(\rho\, \boldsymbol{u}_j \boldsymbol{u}_{i,j}) + \boldsymbol{w}_{ns,i}\, \boldsymbol{\sigma}_{ij} - \boldsymbol{w}_{ns}\, \boldsymbol{f}_i]\, d\Omega + \Sigma_e \int_{\Omega_e} [\boldsymbol{p}_{ns}(\rho\, \boldsymbol{u}_j \boldsymbol{u}_{i,j}) - \boldsymbol{p}_{ns}\, \boldsymbol{\sigma}_{ij,j} - \boldsymbol{p}_{ns}\, \boldsymbol{f}_i]\, d\Omega$$
$$- \Sigma_e \int_{\Gamma_{int}} \boldsymbol{w}_{ns} [\boldsymbol{\sigma}_{ij}\, \boldsymbol{n}_i]\, d\Gamma - \int_{\Gamma_h} \boldsymbol{w}_h\, \boldsymbol{h}_i\, d\Gamma = 0$$

or

$$\Sigma_e \int_{\Omega_e} \boldsymbol{w}'_{ns} [(\rho\, \boldsymbol{u}_j \boldsymbol{u}_{i,j}) - \boldsymbol{\sigma}_{ij,j} - \boldsymbol{f}_i]\, d\Omega - \Sigma_e \int_{\Gamma_{int}} \boldsymbol{w}_{ns} [\boldsymbol{\sigma}_{ij}\, \boldsymbol{n}_i]\, d\Gamma - \int_{\Gamma_h} \boldsymbol{w}_h (\boldsymbol{\sigma}_{ij}\, \boldsymbol{n}_i - \boldsymbol{h}_i)\, d\Gamma = 0$$

- from the integral problem is again possible to *extract* the orginal differential form

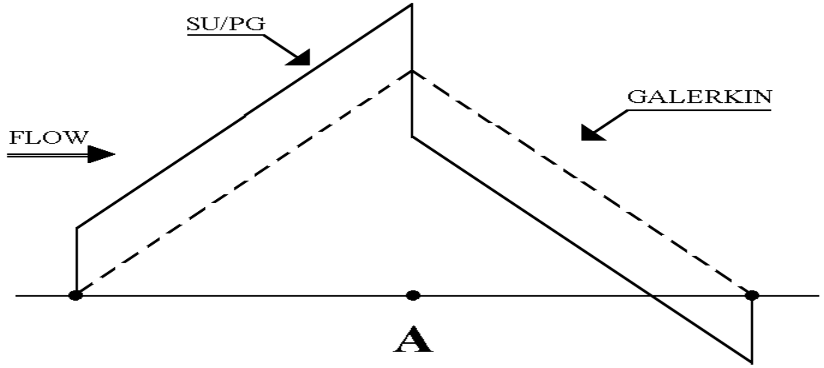
$$\rho\, \boldsymbol{u}_j \boldsymbol{u}_{i,j} + \boldsymbol{\sigma}_{ij,j} - \boldsymbol{f}_i = 0$$
$$(\boldsymbol{\sigma}_{ij}\, \boldsymbol{n}_i - \boldsymbol{h}_i) = 0 \cdot \rightarrow \cdot \Gamma_h$$
$$[\boldsymbol{\sigma}_{ij}\, \boldsymbol{n}_i] = 0 \cdot \rightarrow \cdot \Gamma_{int}$$

Euler - Lagrange conditions of residual FEM formulation

CONSISTENT *STREAMLINE UPWIND-PETROV GALERKIN* FEM FORMULATION (3)

- the expression of *streamline upwind* perturbation $\boldsymbol{p}_{ns}$ is

$$p_{ns} = \tilde{k}\, \bar{\boldsymbol{u}}_j\, w_{ns},_j / \|\boldsymbol{u}\|$$



- the artificial diffusivity for multi-dimensional cases is defined as

$$\tilde{k} = \frac{\left(\bar{x}\, u_x\, h_x + \bar{h}\, u_h\, h_h + \bar{V}\, u_V\, h_V\right)}{2}$$

$$x = coth(a_x) - 1/a_x$$

$$h = coth(a_h) - 1/a_h$$

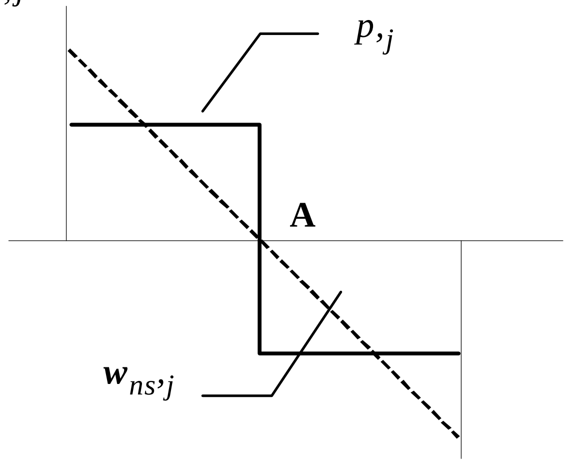
$$V = coth(a_V) - 1/a_V$$

THE XENIOS *STREAMLINE UPWIND-PETROV GALERKIN*

- two fundamental changes has been introduced with respect to SUPG formulation (Brooks&Hughes, 1982)

$$\int_{W_e} p_{ns} s_{ij,j} dW = -\int_{W_e} p_{ns} \; p_{,j} dW + \int_{W_e} p_{ns} \; (2k_{ij}(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}))_{,j} dW$$

- $\int_{W_e} \tilde{k} \frac{\bar{u}_j}{\|u\|} w_{ns,j} \; p_{,j} dW$



- artificial elementary fluxes have to be explicitly balanced along physical domain bounadries

$$\int_{W_e} p_{ns} s_{ij,j} dW = -\int_{W_e} p_{ns,j} s_{ij} dW + \int_{G_e} p_{ns} \; s_{ij} \boldsymbol{n}_j dW \;\Rightarrow\; \int_{W_e} \boldsymbol{d}_b p_{ns} s_{ij,j} dW$$

STABILIZATION OF DIFFUSION DOMINATED FLOWS

- the relaxing of incompressibility constraint to velocity field is mandatory

  the original non-consistent approach (Brezzi&Pitkaranta, 1984) set $u_{i,i} \propto (p_{,i})_{,i}$

- for *Stokes* problem a consistent stabilization leads to a *Petrov-Galerkin residual* method

  with perturbed Stokes weight of the form $\boldsymbol{w}'_s = \boldsymbol{w}_s + \alpha_e h_e^2 w_{c,i}$

$$\sum_e \int_{\Omega_e} [\boldsymbol{w}_{s,i}\,\boldsymbol{\sigma}_{ij} + \boldsymbol{w}_s \boldsymbol{f}_i]\,d\Omega + \sum_e \int_{\Omega_e} \alpha_e h_e^2\, w_{c,i} [\boldsymbol{\sigma}_{ij,i} + \boldsymbol{f}_i]\,d\Omega +$$
$$-\sum_e \int_{\Gamma_{int}} \boldsymbol{w}_s [\boldsymbol{\sigma}_{ij} \boldsymbol{n}_i]\,d\Gamma - \int_{\Gamma_h} \boldsymbol{w}_h \boldsymbol{h}_i\,d\Gamma + \sum_e \int_{\Omega_e} w_c\,\boldsymbol{u}_{i,i}\,d\Omega = 0$$

- the stabilizing integral contains the term

$$\int_{\Omega_e} \alpha_e h_e^2 w_{c,i}\, p_{,i}\,d\Omega = \int_{\Omega_e} \alpha_e h_e^2 w_c (p_{,i})_{,i}\,d\Omega - \int_{\Gamma_e} \alpha_e h_e^2 w_c\, p_{,i} \boldsymbol{n}_i\,d\Gamma$$

CONSISTENT *PRESSURE STABILIZED - PETROV GALERKIN* FEM FORMULATION

- a general stabilization technique for the incompressibility constraint of Navier-Stokes flows

  acts by introducing a perturbation of the *continuity equation* (Tezduyar, 1992)

$$\sum_e \int_{\Omega_e} \frac{1}{\rho} \tau_{pspg} w_{c,i} \left[\rho \mathbf{u}_j \mathbf{u}_{i,j} - \sigma_{ij,j} - \mathbf{f}_i \right] d\Omega$$

  the stabilization factor is $$\tau_{pspg} = \frac{h}{2U} \gamma (Re^U)$$

- the relaxation of the incompressibility constraint is made proportional to the error affecting the velocity field solution

COMPARATIVE ANALYSIS OF STABILIZATION EFFECTS (1)

	Schemes	Grid	Coefficients l_p	Coefficients l_c	n_{it}	$\frac{(\dot{m}_{in} - \dot{m}_{out})}{\%(\dot{m}_{in})}$	$\frac{(P^{av}_{0_{in}} - P^{av}_{0_{out}})}{\%(Q_{in})}$
Borello, Corsini and Rispoli, 2000 Petrov-Galerkin Schemes	PG11	C	1	1	130	1.7×10^{-1}	1.8×10^{-2}
	PG21	**C**	**1**	**1**	**72**	$\mathbf{8.7\times10^{-3}}$	$\mathbf{2.1\times10^{-4}}$
	IPG21	C	0.05	1	86	8.8×10^{-3}	8.7×10^{-2}
	PG11	M	1	1	445	8.4×10^{-4}	3.4×10^{-4}
	PG21	**M**	**1**	**1**	**110**	$\mathbf{1.4\times10^{-4}}$	$\mathbf{4.0\times10^{-5}}$
	IPG21	M	0.05	1	202	3.9×10^{-4}	2.2×10^{-2}
	PG11	F	1	1	437	1.7×10^{-4}	9.6×10^{-5}
	PG21	**F**	**1**	**1**	**170**	$\mathbf{2.6\times10^{-5}}$	$\mathbf{2.6\times10^{-6}}$
	IPG21	F	0.3	1	204	5.85×10^{-5}	5.5×10^{-3}

	Schemes	Grid	e_2	e_4	e_{pw}	n_{it}	$\frac{(\dot{m}_{in} - \dot{m}_{out})}{\%(\dot{m}_{in})}$	$\frac{(P^{av}_{0_{in}} - P^{av}_{0_{out}})}{\%(Q_{in})}$
Basson and Lakhsminarayana, 1994 4th order artificial dissipation scheme	FV	C	0.0	0.25	0.0	-	1.1×10^{-1}	-2.0×10^{-2}

e_2 and e_4 are respectivelly the coefficient for 2nd and 4th order artificial dissipation, e_{pw} is the coefficient for pressure weighting

Grids (x, h)

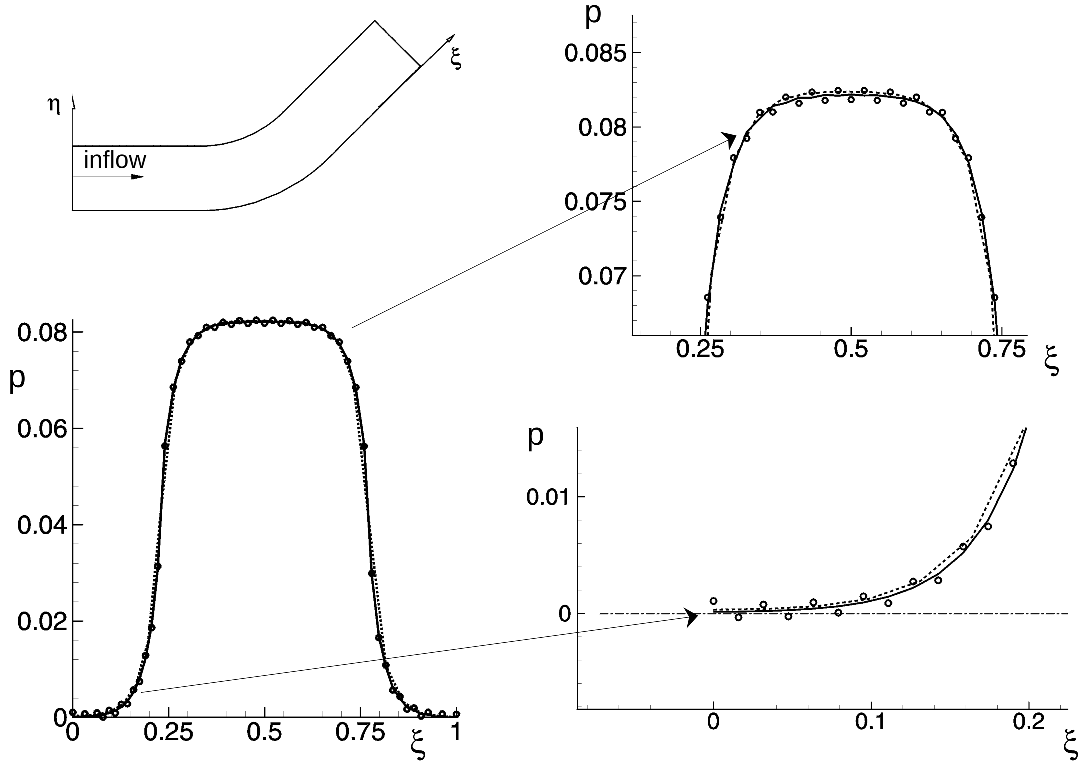
C, (53×15)

M, (105×31)

F, (209×63)

INVISCID FLOW IN 45° TWO-DIMENSIONAL BEND

COMPARATIVE ANALYSIS OF STABILIZATION EFFECTS (2)



Computed pressure distributions on bend pressure surface, (M) grid

symbols: Galerkin solution; solid lines: PG21 solution; dashed lines: PG11 solution