<u>**Technical Acoustics and Noise Control**</u> *(lecture notes for self-learning)*

Dr. Gábor KOSCSÓ titular associate professor (BME Department of Fluid Mechanics)

Lecture 4.

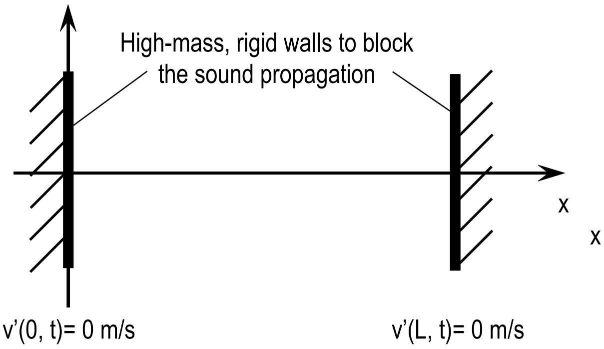
**Content:**

4.1. The solution of the homogeneous acoustic wave equation in enclosed space (lecture notes)

4.2. Test questions and solved problems

### 4.1. The solution of the homogeneous acoustic wave equation in enclosed space

The general solution and the harmonic partial solution of the wave equation describe outdoor sound propagation, in which case there is nothing blocking the linear sound propagation, no reflections are formed. This condition is well suited for solving a number of acoustic problems (e.g., determining the noise in front of a building, caused by a fan operating in the open air at the roof of an adjacent building). The other big part of the acoustic problems concerns an enclosed space bounded by walls that block propagation. The characteristics of the sound field in enclosed space created by the sound source change significantly, compared to the outdoor case (in addition to direct radiation, reflections must also be taken into account). In a bounded space, the wave acoustic model draws our attention to other important phenomena, so it is worth examining the problem in detail. As before, the solution of the wave equation for a finite space is examined, in a first step for a one-dimensional case. To do this, place two high-mass, rigid (non-deformable by force), airtight (non-porous, no holes) walls that are non-permeable for sound perpendicular to the axis at distances x= 0 m and x= L m along the x-axis, see figure.




Blocking the sound propagation with rigid, high-mass, airtight walls

According to the no-slip condition in fluid mechanics (the relative velocity between a solid surface and the adjacent liquid layer is zero), if the boundary walls are at static rest, the velocity of the neighbouring liquid layer is also zero, boundary conditions for particle velocities at x = 0 m and x = L m,

$$v'(0, t) = 0 \text{ m/s} \qquad \text{and} \qquad v'(L, t) = 0 \text{ m/s}$$

The presence of walls does not affect the flow nature of sound and the mathematical model associated with it. The continuity theorem, the law of motion, the energy balance, and the ideal gas law are also satisfied in enclosed space, so we start from the wave equation when creating the mathematical model. The boundary conditions can be easily given for the particle velocity, so the wave equation for the particle velocity is used to determine the bounded space solution,

$$\frac{1}{a^2}\frac{\partial^2 v'}{\partial t^2} - \frac{\partial^2 v'}{\partial x^2} = 0$$

The general solution of the wave equation,

$$v'(x,t) = f(at - x) + g(at + x)$$

Let us examine how the boundary conditions change the general solution. Substituted into the general solution at x= 0 m,

$$v'(0,t) = 0 = f(at - 0) + g(at + 0)\ , \text{ rearranging, } f(at) = -g(at)$$

In place of the component function “f”, write “- g” and substitute for x= L m,

$$v'(L,t) = 0 = f(at - L) + g(at + L) = -g(at - L) + g(at + L)$$

Shifting the function “g” by (-L) and rearranging, it can be seen, that the function “g” is periodic according to 2L distance,

$$g(at) = g(at + 2L)$$

In summary, the boundary conditions limit the general solution, instead of being arbitrary, the components “f” and “g” are identical but have opposite signs and are periodic according to 2L. The Fourier series of the periodic function “g”,

$$g(at \pm x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left[\alpha_n cos\frac{2\pi n}{2L}(at \pm x) + \beta_n sin\frac{2\pi n}{2L}(at \pm x)\right]$$

The components of each Fourier series n values are physically harmonic wave components, so the 2L period along the length is actually the wavelength (λ). Thus coefficients of the bracketed term in the arguments of the cosine and sine functions, the ratio 2π/2L is the wavenumber (k= 2π/λ),

$$\frac{2\pi n}{2L} = \frac{2\pi}{\lambda}n = k \cdot n = k_n\ , \quad \text{where n= 1, 2, 3, … (natural numbers)}$$

To determine the solution of the wave equation valid in a finite space, substitute the appropriately signed members of the Fourier series and apply the new notation,

$$v'(x,t) = f(at - x) + g(at + x) = -g(at - x) + g(at + x) =$$

$$= -\frac{\alpha_0}{2} - \sum_{n=1}^{\infty}[\alpha_n cosk_n(at - x) + \beta_n sink_n(at - x)] +$$

$$+\frac{\alpha_0}{2} + \sum_{n=1}^{\infty}[\alpha_n cosk_n(at + x) + \beta_n sink_n(at + x)]$$

Due to the opposite sign, the equilibrium term $\alpha_0/2$ is lost, the physical meaning of which is that due to the presence of boundary walls, the equilibrium value of the sound field variables (e.g., equilibrium pressure) does not change. By aggregating the two series and removing the coefficients,

$$= \sum_{n=1}^{\infty}\left[\alpha_n\big(cosk_n(at + x) - cosk_n(at - x)\big) + \beta_n\big(sink_n(at + x) - sink_n(at - x)\big)\right]$$

Using the identities for resolving the sine and cosine of the angle sum and difference,

$$v'(x,t) = \sum_{n=1}^{\infty}[-2\alpha_n sink_n at\ sink_n x + 2\beta_n cosk_n at\ sink_n x]$$

The $k_n a$ product is the angular frequency ($\omega_n$) of the n[th] harmonic component, and removing location-dependent sinusoidal term from the brackets, the solution of the on-dimensional homogeneous acoustic wave equation in enclosed space is,

$$v'(x,t) = \sum_{n=1}^{\infty}[sink_n x(-2\alpha_n sin\omega_n t + 2\beta_n cos\omega_n t)]$$

Where: $k_n = \frac{2\pi}{2L}n$ , and $\omega_n = k_n a = \frac{2\pi}{2L}na$

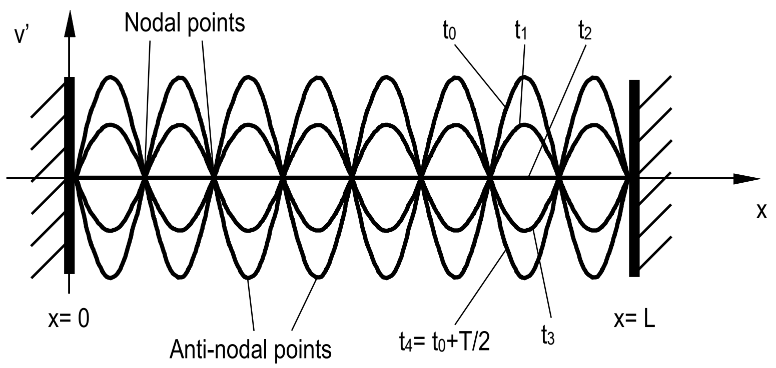
**Comments:**

- The most important difference between the general and bounded space solutions of the wave equation is that in the argument of the function describing the wave, the place and time variables are included together in the general solution, in the bounded space solution separately. The argument (t±x/a) refers to the propagating nature of the wave. In the bounded space solution, the separation of the variables x and t indicates the give up of the propagating nature of the wave. The presence of the two walls actually, physically restricts the free propagation of sound, and this fact is expressed, in the specific language of mathematics (by the separation of variables).

- Let take the $n_{th}$ element of the series of the previous solution, and let $-2\alpha_n = \hat{v}$ ,and let $\beta_n = 0$,

$$v'_n(x,t) = \hat{v}\ sin\omega_n t\ sink_n x$$

The particle velocity between the two walls at all points where $k_n x$ = 0, π, 2π, 3π,… (of course also at x= 0 and x= L) regardless of time is 0 m/s, these are the nodal points of the wave. Offset by π/2 radians, where $k_n x$ = π/2, 3π/2, 5π/2,… the amplitude of the wave takes a maximum value, these are the anti-nodal point of the wave. Another important difference is that in the section between two nodes all liquid parts move in the same phase, i.e. they move to the right or to the left uniformly, only their amplitude differs from each other. The propagating wave in the presence of the two walls turned into an oscillation of the flexible, continuous medium, characterized by a periodic system of nodes and anti-nodes, see figure.




Particle velocity distribution in a wave field formed between two walls at different times ($t_0$, $t_1$, …)
(the time $t_4$ is half a period later than $t_0$)

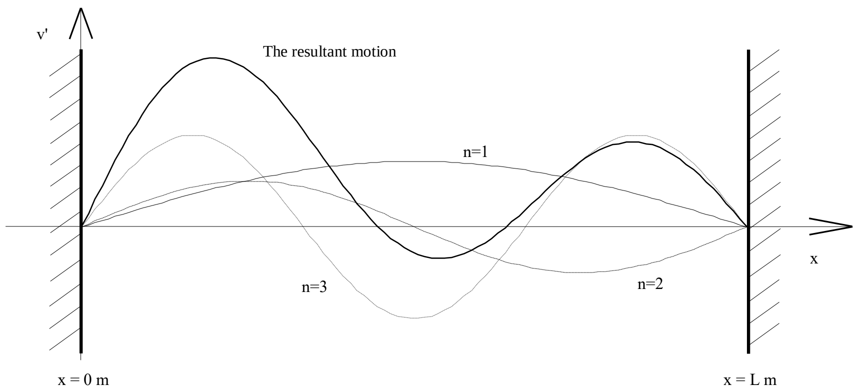
- The Fourier coefficients $\alpha_n$ and $\beta_n$ can be determined from the distribution of the particle velocity v' at the initial time (t= 0), from the initial value condition.

- In mathematical language, the constants $\omega_1$, $\omega_2$, $\omega_3$,… and $k_1$, $k_2$, $k_3$,…, are the eigenvalues of the problem. The $\omega_1$ is the fundamental harmonic (or fundamental angular frequency) of the system, the other $\omega_2$, $\omega_3$,… are upper harmonics. The values of ω and k refer to the case of a specific acoustic arrangement (distance between walls, speed of sound,…). Substituting the eigenvalues into the bounded-space solution functions we get the eigen-functions,

$$v'_1(x,t) = sink_1x(-2\alpha_1 sin\omega_1 t + 2\beta_1 cos\omega_1 t)$$
$$v'_2(x,t) = sink_2x(-2\alpha_2 sin\omega_2 t + 2\beta_2 cos\omega_2 t)$$
$$v'_3(x,t) = \cdots$$

which satisfy both the wave equation and the connecting boundary conditions.

- It is very important to realize that eigen-functions are only possibilities in the system. What is heard in the sound field depends fundamentally on the excitation. If someone talking in a space bounded by thick walls, the voice of the speaker will basically be heard in the sound field. The eigen-functions can be “pick-up” in the sound field by creating a pulsed (short-term) sound generation (e.g., applause), an initial disturbing state (initial condition), which causes the particles to move in the sound field. The system left alone after the initial perturbation then performs its own vibrations, such as the tensioned and then released mass-spring one-degree of freedom vibration system, or the tensioned guitar string. (While the one-degree of freedom vibrating system oscillates at one eigen-frequency, the air column enclosed between two walls and the string stretched between the two points, like a finite piece of continuous medium, can oscillate at any number of discrete frequencies.) The eigen-function, closely related to the previous explanation, can be derived in this aspects too, since the system left after the initial, pulsed excitation describes its own motion (-vibration). The particle velocity distribution as a function of space at a given moment of the first three components (base frequency and adjacent two harmonics) and the resulting motion created by a suitable initial sound generation is shown on the next figure,




Distribution of the particle velocities of the base and the first two upper harmonics and the resulting wave in a simple sound field as a function of location at a fixed time

- In a bounded space, the solution of the wave equation was determined for the particle velocity variable due to the boundary conditions. From an experimental (measurement) point of view, knowledge of sound pressure can also be important. In order to determine the sound pressure function, let the function describing the velocity field be for n= 1 (base frequency), the indices 1 are omitted for simplification.

$$v'(x,t) = \hat{v}\, sin\omega t\; sinkx$$

The linear acoustic motion equation creates a relationship between sound pressure and particle velocity. Expressing the sound pressure from the equation of motion and substituting v’ function,

$$p'(x,t) = -\rho_0 \int \frac{\partial v'}{\partial t} dx = -\rho_0 \int \frac{\partial}{\partial t}(\hat{v}\, sin\omega t\, sinkx) dx = \rho_0 \frac{\omega}{k} \hat{v}\, cos\omega t\, coskx =$$

$$= \rho_0 a \hat{v}\, sin\left(\omega t + \frac{\pi}{2}\right) sin\left(kx + \frac{\pi}{2}\right) = \hat{p}\, sin\left(\omega t + \frac{\pi}{2}\right) sin\left(kx + \frac{\pi}{2}\right)$$

In the sound pressure function, the sound pressure amplitude is the same as the equation of motion already derived in the algebraic relationship system (the higher-level, place- and time-dependent mathematical model includes the algebraic description). The shape of the space- and time function and the coefficients x and t (ω and k) for particle velocity and sound pressure are the same (sound field variables change simultaneously). By rewriting the cosines of the sound pressure wave function to sines, it is easy to see that there is a π/2 phase shift difference (quarter period) between the particle velocity and the sound pressure, both the space and time variable. (For free-propagating plane waves, there is no phase difference between particle velocity and sound pressure.)

- The vibration of a finite long air column enclosed between two walls can be illustrated by the axial vibration of a coil spring placed between the walls and rigidly attached to the walls at both ends. By pulling the coil spring in the middle in the axial direction to the left and then abruptly releasing it, the fundamental harmonic motion can be divided into four characteristic sections. In the first stage, the center of the spring travels to the right at maximum speed, in this time the spring is tension free. A quarter of a period later, the movement of the spring compressed to the right stops for a moment, a maximum compressive force is generated on the right end and a maximum tensile force is generated on the left end. Another quarter later, the spring is tensionless again and the center is to the left at maximum speed. In the last quarter, the movement of the spring compressed to the left stops again for a moment, with a maximum tensile force on the right end and a compressive force on the left end. A similar movement is performed by the air column bounded by walls on both ends (in the sound space, the force corresponds to the sound pressure, the speed to the particle velocity variable).

- A question arises as, what is the importance in engineering practice of the bounded space solution of the wave equation, the eigen-functions and eigen-vibrations. Liquid is transported in pipe several times. The fluid in the pipeline can be take as a finite long piece of flexible, continuous medium capable of performing its own vibrations at given frequencies. If the motion of the medium in the tube is driven by a positive displacement pump, the transported medium is excited at a given frequency due to the periodic operation. If the excitation frequency is equal to one of the eigen-frequencies, a resonance is formed. The high-amplitude motion generated during resonance causes a significant mechanical load in the system, which can cause abnormal operation and, ultimately, failure. It is important to note that small amplitude waves were assumed during derivation. During resonance, the magnitude of the amplitude can increase significantly, making the accuracy of the linear model worse. Knowledge of the dynamic behaviour of pipelines is basic and important mechanical engineering knowledge, and the bounded space solution of the wave equation provides introductory course for this. Resonance is an important phenomenon in acoustics, so we will deal with it in more detail later.

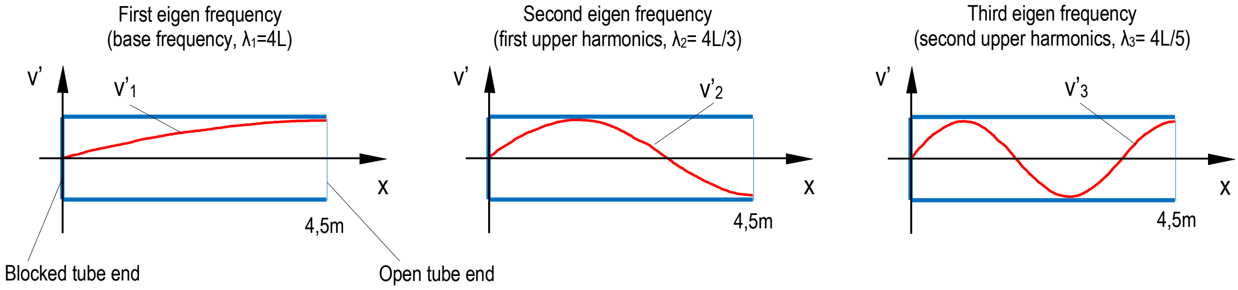
- 2 and 3 dimensional oscillations can occur in a finite-sized, continuous elastic medium, depending on the shape and the nature of the excitation. A fundamental problem in room acoustics is to reduce the effect of amplifications and attenuations due to airborne natural modes in the rooms. But multi-dimensional continuum vibrations can occur not only in liquids but also in solid flexible material such as the metal shell of a bell or a gear.

## 4.2. Test questions and solved problems

S.P.1. Let calculate the first and third acoustic natural frequencies of the air column in a 4.5 m long tube closed at one end and open at the other end. The air temperature (t) 25 °C.

$$a = \sqrt{\kappa R T_0} = \sqrt{1{,}4 \cdot 287 \cdot (273 + 25)} \approx 346\ m/s$$

The solution is a harmonic continuum vibration, shown in the previous section, based on the solution of the wave equation in the bounded space. Harmonic vibrations are described by sine and cosine functions. Boundary conditions can be easily given for particle velocity from a physical point of view. At closed tube ends, the particle velocity is zero. If the closed end of the tube is at rest, the fluid layer just adjacent to it also stands according to the no-slip condition in fluid mechanics. There is nothing effecting the motion at the open tube ends, so the particle velocity is maximum here. For a given pipe section, the wavelength of the self-oscillation is appropriate if the magnitude of the sine is zero for a closed pipe end and the maximum of the sine for an open pipe end (regardless of the sign). In our case, it can be seen for the graphical derivation of the solution, the particle velocity distribution of the eigen-vibrations coincides with zero at the closed end of the tube and with the largest absolute value of the sine function at the open end. The lowest frequency has the largest wavelength or the least sinusoidal period, the increasing frequency decreases the wavelength, the number of sinusoidal periods on the pipe section increases, see Fig.




Particle velocity distribution along the x direction of a continuum oscillation in a tube, when closed at one end and open at the other end, for the first three eigen-frequencies

First eigen-frequency: Based on the boundary conditions, the particle velocity distribution graph is the quarter sine period from the minimum to the adjacent maximum. The wavelength and frequency:

$$\lambda_1 = 4 \cdot l/1 = 4 \cdot 4{,}5 = 18 \ m$$

$$f_1 = a/\lambda_1 \approx 346/18 \approx 19{,}2 \ Hz$$

Third eigen-frequency: Based on the boundary conditions, the particle velocity distribution graph is the one and a quarter sine period from the minimum to the second positive maximum. The wavelength and frequency:

$$\lambda_3 = 4 \cdot l/5 = 4 \cdot 4{,}5/5 = 3{,}6 \ m$$

$$f_3 = a/\lambda_3 \approx 346/3{,}6 \approx 96{,}1 \ Hz$$

-----