"nonlinear pendulum"

$\ddot{x} + \sin x = 0 \Rightarrow \left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x \end{aligned}\right\}$ fixed point $(0,0)$, $(0, n\pi)$

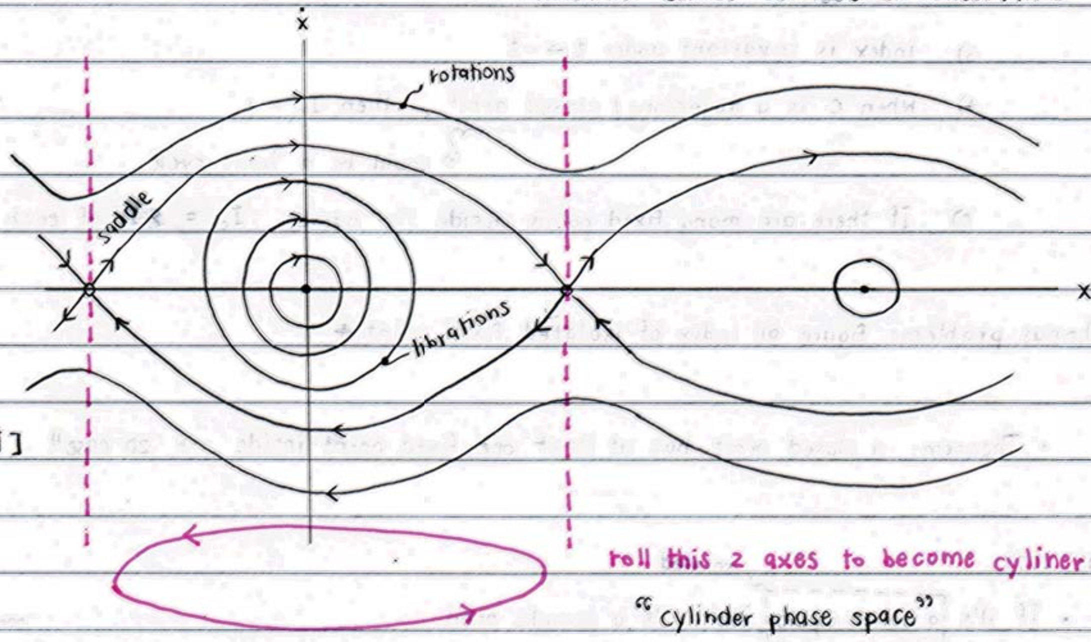
→ Earlier, we linearized this by $\sin x \approx x$. It's time to consider $\sin x$.

$$J = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}_{fp}$$

This system is conservative

- physical: it swings back and forth $\quad \dot{x}\ddot{x} + \dot{x}\sin x = 0$
- numerical: multiply $\dot{x}$ then integrate $\dashrightarrow$ $\frac{1}{2}\dot{x}^2 - \cos x = C$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ $\dashrightarrow$ tell that this is a linear center which is nonlinear center because of conservative



$x \in [-\pi, \pi]$

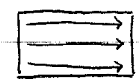
roll this 2 axes to become cyliner

"cylinder phase space"

- We may think that a saddle point is a part of heteroclinic orbit because of 2 pts. connection but it becomes homoclinic orbit on the cylinder "index theory"

- global consideration -- don't tell the stability of fixed points, tell that how many fixed points there are and what type of fixed point.

- I_c is negative when collecting clockwise
  - > stable / unstable node $I_c = 1$
  - > saddle point $I_c = -1$

- properties of the index
  0) integer - valued
  1) $c \to c'$ then $I_c = I_{c'}$
     continuously change, going through a fixed point
  2) when there's no fixed point inside → flow box 
     then $I_c = 0$
  3) index is invariant under $t \to -t$
  4) when C is a trajectory / closed orbit, then $I_c = 1$
     ↳ could be a limit cycle
  5) If there are many fixed points inside the curve $I_c = \Sigma I_i$ of each fixed point

☆ bonus problem: figure all index of isolated fixed point ☆

- Theorem: a closed orbit has at least one fixed point inside --> 2D case !!

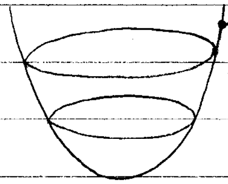
October 13, 2009

- If it's $\boxed{\text{a limit cycle}}$ (isolated), then it's a periodic orbit.
  But if it's a periodic orbit, it's not necessary to be a limit cycle, could be $\boxed{\text{a center}}$ (non-isolated)

- There's a limit cycle in 2D and also in higher-dimensional system

- no limit cycle in conservative system.

- Gradient Sys. -- $\ddot{x} + \nabla V(x) = 0$ — conservative sys = energy conserved



centers: for energy function, circle at constant value

- need smooth vector field for existence and uniqueness solutions

- contraction mapping: $L\|x-y\| \geq \|f(x) - f(y)\|$ where $0 < L \leq 1$

  ↳ Lipschitz continuous

- if we start on the object, we will stay on the object --→ invariant

☆ bonus problem: plot cycle graph in prob 7.3.11 in cylinder coordinate system ☆

- cycle graph is invariant.

- Liapunov stability --→ If we can't find Liapunov function, tells nothing.

- Liapunov gives global info.

  Attracting point gives only local info --→ for linear sys: local is already global.

- Poincaré - Bendixson: can have several closed orbits in trapping region

  ↳ invariant

- Lienard system: $\ddot{x} + f(x)\dot{x} + g(x) = 0$ --→ Lienard plane

  nonlinear in both damping and stiffness

  ex. van der Pol -- $\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$

  Q: what happen if this is $x+1$ instead?

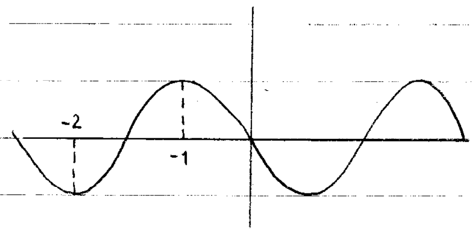
  A: like shift fp... introduce new variable $y = x+1$

- Is there fp in non-autonomous system? -- No!

$\dot{x} = f(x,t) \dashrightarrow \dot{x} = f(x,y)$, $\underbrace{\dot{y}}_{t} = 1$ } no fp because $\dot{y}$ never be zero; also no limit cycle

- nullcline is not a trajectory of the system but help to easily plot phase portrait
  ↳ that's why it doesn't correspond to eigendirection

- x_A, x_B are local minimum and local maximum



⇒ about relaxation oscillator

October 15, 2009

"weakly nonlinear oscillators" $\dashrightarrow \ddot{x} + x + \varepsilon h(x_0, \dot{x}) = 0$ — $\varepsilon \ll 1$ when $\varepsilon = 0$, simple harmonic (undamped)

$\neq$ relaxation oscillator -- $\mu \gg 1$

ex. duffing oscillator $\ddot{x} + 2\xi\dot{x} + x + \gamma x^3 = F\cos\Omega t$ — $\gamma \ll 1$

Let's consider $\ddot{y} + 2\xi\dot{y} + y = F\cos(\Omega t + \phi_0)$; $y(0) = y_0$, $\dot{y}(0) = V_0$

Ω: excitation frequency, ϕ_0: phase

Free oscillations: $\ddot{y} + 2\xi\dot{y} + y = 0$ (y_0, V_0)

$y = ce^{\lambda t} \dashrightarrow$ characteristic eq. $\lambda^2 + 2\xi\lambda + 1 = 0$

$\lambda_{1,2} = -\xi \pm \sqrt{\xi^2 - 1}$ — both negative

$D = 4(\xi^2 - 1)$ ① overdamped oscillator $\xi > 1$, $D > 0$

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$ --→ decay w/ no oscillation

$$c_1 = \frac{y_0}{2}\left(1 - \frac{1}{\sqrt{\xi^2-1}}\right) - \frac{v_0}{2\sqrt{\xi^2-1}}$$

$$c_2 = \frac{y_0}{2}\left(1 + \frac{1}{\sqrt{\xi^2-1}}\right) + \frac{v_0}{2\sqrt{\xi^2-1}}$$

② $\xi = 1$, $D = 0$: $y(t) = y_0 e^{-t} + (v_0 + y_0) t e^{-t}$ -- critical

③ underdamped $\xi < 1$, $D < 0$

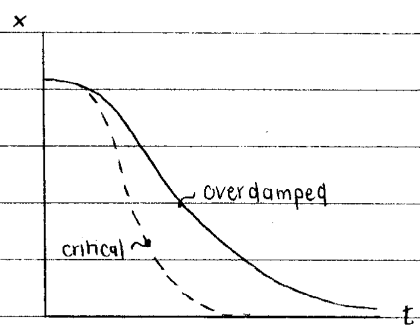
natural freq. was modified

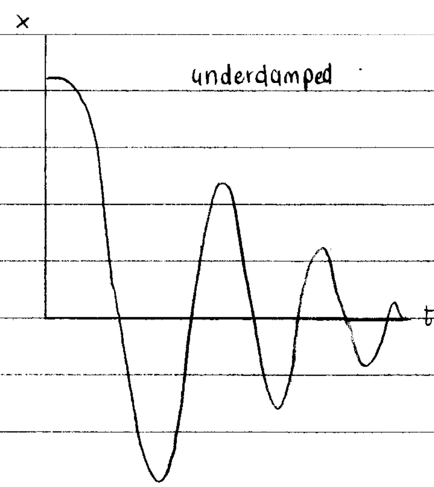
$$y(t) = e^{-\xi t}\left[y_0 \cos(\sqrt{1-\xi^2}\, t) + \frac{v_0 + \xi y_0}{\sqrt{1-\xi^2}} \sin(\sqrt{1-\xi^2}\, t)\right]$$

$$= c e^{-\xi t} \sin(\omega_d t + \ell)$$

$$c = \sqrt{y_0^2 + \left(\frac{v_0 + \xi y_0}{\omega_d}\right)^2}, \quad \ell = \tan^{-1}\left(\frac{y_0\sqrt{1-\xi^2}}{v_0 + \xi y_0}\right)$$

two timescales!!





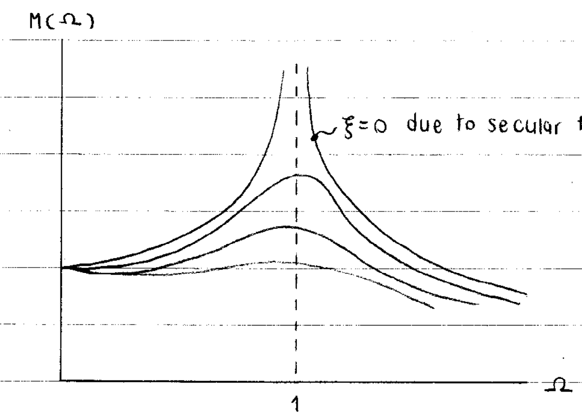
nondimensional $\Omega = \frac{\omega}{\omega_n}$

forced oscillators: $\ddot{y} + 2\xi\dot{y} + y = F\cos\Omega t$ (y_o, v_o) --> linear, so can do superposition with steady state solution and transient state solution (homogeneous and particular)

$y(t) = \underbrace{Ce^{-\xi t}\cos(\omega t+\ell)}_{\text{disappear when } t\to\infty} + A\cos(\Omega t - \phi)$

when not resonant $\Omega \neq 1$

$$A = M|F| \; ; \quad M(\Omega) = \frac{1}{\sqrt{(1-\Omega^2)^2 + 4\xi^2\Omega^2}} > 0$$



when $\Omega = 1$: resonant excitation $y_{inhom}(t) = C_1\cos(t+\phi) + \underbrace{C_2 t\sin t}_{\text{secular term}}$

need these knowledge to understand perturbation theory

$\ddot{x} + 2\varepsilon\dot{x} + x = 0$; $x(0) = 0$, $\dot{x}(0) = 1$

regular perturbation theory: $x(t) = x_o(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$

know from linear analysis when $\varepsilon = 0$

plug back to $\ddot{x} + 2\varepsilon\dot{x} + x = 0$

assume that we can write a solution as series like above and then check later (need to consider about convergence also)

solution is $X(t,\varepsilon)$ -- we know $X(t,0) = x_o(t)$ then we use perturbation theory to help us write $X(t,\varepsilon)$

$$\frac{d^2}{dt^2}(x_0 + \varepsilon x_1 + \dots) + 2\varepsilon \frac{d}{dt}(x_0 + \varepsilon x_1 + \dots) + x_0 + \varepsilon x_1 + \dots = 0$$ --> then group terms with the same power in ε.

$x^2 + 2\varepsilon x - 1$ -- regular perturbation
$\varepsilon x^2 + 2x - 1$ -- singular perturbation

$\mathcal{O}(1)$ -- ε^0 : $\ddot{x}_0 + x_0 = 0$ $\quad x_0(0) = 0, \quad \dot{x}_0(0) = 1$

solution is $x_0(t) = \sin t$

$\mathcal{O}(\varepsilon)$ -- ε^1: $\ddot{x}_1 + 2\dot{x}_0 + x_1 = 0$ --> we already know x_0, move it to RHS as forcing

$$\underbrace{\ddot{x}_1 + x_1}_{\omega_n = 1} = \underbrace{-2\cos t}_{\omega = 1} \qquad x_1(0) = 0, \quad \dot{x}_1(0) = 0$$

resonant!!

$$x_1 = -t\sin t \quad \text{(secular)}$$

☆ bonus problem: do the $\mathcal{O}(\varepsilon^2)$ solution ☆

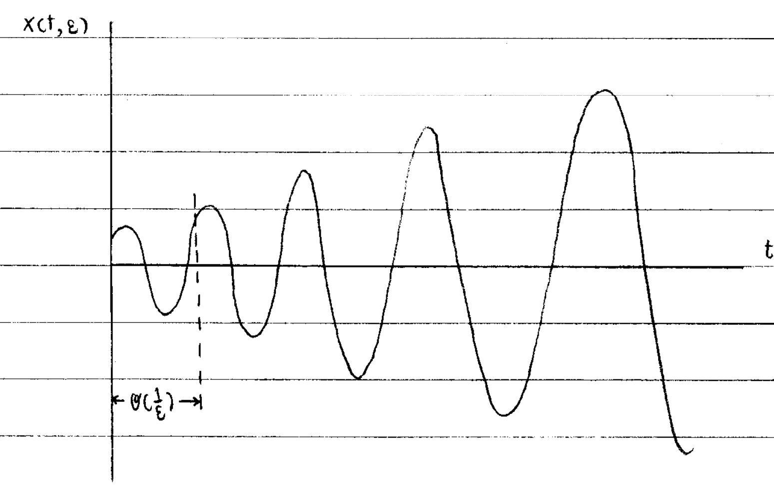
$\left.\begin{array}{l} x_0(0) = 0, \quad \dot{x}_0(0) = 1 \\ x_1(0) = 0, \quad \dot{x}_1(0) = 0 \end{array}\right\}$ get this when expand as series also

$$x(0) = 0 = x_0(0) + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \dots$$

$$\dot{x}(0) = 1 = \dot{x}_0(0) + \varepsilon \dot{x}_1(0) + \varepsilon^2 \dot{x}_2(0) + \dots$$

$\dot{x}_1(0) = 0$!!

Summary: $x(t,\varepsilon) = \sin t - \varepsilon t \sin t + \mathcal{O}(\varepsilon^2)$



October 20, 2009

"Method of Multiple Scales (MMS)"

$$\ddot{x} + x = \varepsilon F(x, \dot{x}) \quad \text{-- weakly nonlinear oscillators}$$

idea: $\ddot{x} + 2\varepsilon\dot{x} + x = 0$

↳ solution $x(t) = ce^{-\varepsilon t}(\underbrace{\sqrt{1-\varepsilon^2}}_{\text{detuned frequency!!}}\, t + \phi)$

two time scales

failure of perturbation method

↓

secular term

seek solution in the form of $x(t) = x_0(t_0, t_1) + \varepsilon x_1(t_0, t_1) + \ldots$ where t_0, t_1 are multiple time scales and are independent

$t_0 = t$

$t_1 = \varepsilon t$ (slow time)

what do we do when perturbation method breaks down? (expand only x) $\dashrightarrow$ Poincaré - Lindstedt (expand x and ω)

~ introduce $\omega = 1 + \varepsilon\sigma$ ($\varepsilon\sigma$: detuning)

~ instead of $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

$x(t, \omega) = \ldots$

~ we have no freedom to just set secular $= 0$, then introduce more degree of freedom that is ω ~ extra freedom to kill secular terms

$$\frac{dx}{dt} = \frac{\partial x_0}{\partial t_0}\frac{dt_0}{dt} + \frac{\partial x_0}{\partial t_1}\frac{dt_1}{dt} + \varepsilon\left[\frac{\partial x_1}{\partial t_0}\frac{dt_0}{dt} + \frac{\partial x_1}{\partial t_1}\frac{dt_1}{dt}\right] + O(\varepsilon^2)$$

$$= \frac{\partial x_0}{\partial t_0} + \varepsilon\frac{\partial x_1}{\partial t_0} + \varepsilon\frac{\partial x_0}{\partial t_1} + O(\varepsilon^2)$$

define differential operators: $D_0 = \frac{\partial}{\partial t_0}$, $D_1 = \frac{\partial}{\partial t_1}$

do on $\underset{\sim}{x} = (x_0, \varepsilon x_1)$

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + O(\varepsilon^2)$$

$$\frac{d^2}{dt^2} = \left(\frac{d}{dt}\right)^2 = (D_0 + \varepsilon D_1 + O(\varepsilon^2))^2 = D_0^2 + 2\varepsilon D_0 D_1 + O(\varepsilon^2)$$

use above with $\ddot{x} + x = \varepsilon F(x,\dot{x})$, we get

ε^0: $D_0^2 x_0(t_0,t_1) + x_0(t_0,t_1) = 0$ --> simple harmonic

ε^1: $D_0^2 x_1(t_0,t_1) + x_1(t_0,t_1) = F(x,\dot{x})$

solve ε^0: $x_0(t_0,t_1) = A(t_1)e^{it_0} + \overline{A(t_1)}e^{-it_0}$

amplitude is slowly varying: t_1 = slow time

substitute $x_0(t_0,t_1) = A(t_1)e^{it_0} + \overline{A(t_1)}e^{-it_0}$ into ε_1 and eliminate secular terms

we'll get differential eq^{n} for $A(t_1) = \frac{1}{2}\alpha(t_1)e^{i\beta(t_1)}$ ← slowly varying phase $\Rightarrow$ $\dot{\alpha} = \ldots$

slowly varying amplitude $\dot{\beta} = \ldots$

do this in the example in the book!!

"KBM (Krylov - Bogoliubov - Mitropolsky)"

$$\ddot{x} + x = F(x,\dot{x},\varepsilon) \quad (*) \qquad \text{when } F=0\text{, simple harmonic}$$

~ if $\varepsilon = 0$; $x(t) = r\cos(t+\phi)$

$\dot{x}(t) = -r\sin(t+\phi)$ $\Rightarrow$ modify this for $\varepsilon \ll 1$

slowly varying

• if $\varepsilon \ll 1$; $x = r(t)\cos(t+\phi(t))$ $\Rightarrow$ let $\theta = t+\phi$

$\dot{x} = \dot{r}\cos\theta - r\sin\theta - r\dot{\phi}\sin\theta$

$r\sin\theta$ same as $\varepsilon = 0$

impose the condition $\dot{r}\cos\theta - r\dot{\phi}\sin\theta = 0$ $(**)$ --> ($\dot{x}$ in $\varepsilon \ll 1$) = ($\dot{x}$ in $\varepsilon = 0$)

s.t. $\dot{x} = -r(t)\sin(\underbrace{t+\phi(t)}_{\theta})$

$$\ddot{x} = -\dot{r}\sin\theta - r\cos\theta - r\dot{\phi}\cos\theta$$

substitute back to $(*)$ and get

$\dot{r}\sin\theta + r\dot{\phi}\cos\theta = -F(r\cos\theta, -r\sin\theta, \varepsilon)$ $(***)$

$(**)\cos\theta + (***)\sin\theta$ --> $\dot{r} = -F(r\cos\theta, -r\sin\theta, \varepsilon)\sin\theta$

$(**)\sin\theta - (***)\cos\theta$ --> $\dot{\phi} = -\frac{1}{r}F(r\cos\theta, -r\sin\theta, \varepsilon)\cos\theta$

average!! because amplitude and phase are slowly varying

$$S(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r\cos\theta, \dots)\sin\theta \, d\theta$$

$$C(r) = \frac{1}{2\pi}\int_0^{2\pi} F(\dots)\cos\theta \, d\theta$$

} Fourier

$$\dot{r} = -S(r)$$

$$\dot{\phi} = -\frac{C(r)}{r}$$

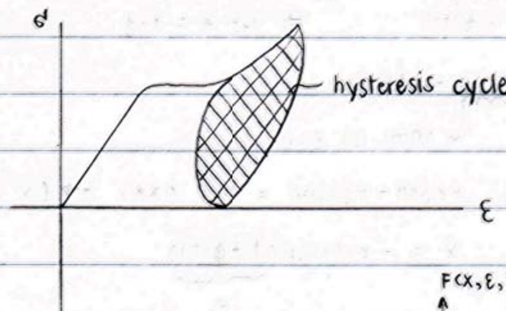
= slow flow equations!! if we can find fp in this sys, then there's a periodic orbit

unforced system

KBM for forced oscillations $\ddot{x} + x = F(x, \dot{x}, \varepsilon) + A\cos\omega t$

$$\boxed{\begin{aligned} -\omega\dot{r} - S(r) &= \frac{A}{2}\sin\phi \\ \frac{r}{2}(1-\omega^2) - \omega r\dot{\phi} - C(r) &= \frac{A}{2}\cos\phi \end{aligned}}$$

hysteresis cycle in material — like add damping to the system

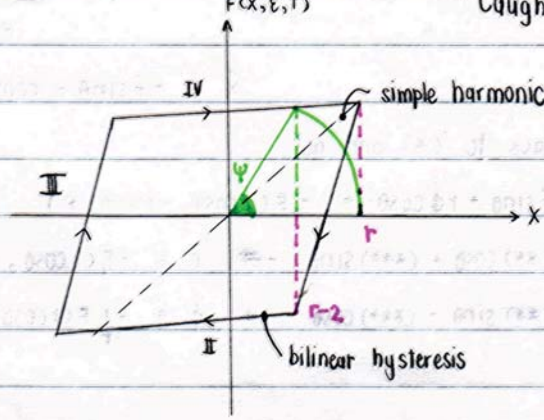


"Caughey (1960) Paper"

$\ddot{x} + x = F(x, \dot{x}, \varepsilon) + A\cos\omega t$

$\ddot{x} + G(x, \varepsilon) = A\cos\omega t$

$F(x, \varepsilon) = x - G(x, \varepsilon)$



⇒ symmetry

max amplitude of steady state solution (oscillation)

$G_I = x - \varepsilon(r_m - 1)$ homogeneous → transient

$G_{II} = (1-\varepsilon)x - \varepsilon$

$G_{III} = x + \varepsilon(r-1)$

$G_{IV} = (1-\varepsilon)x + \varepsilon$

there's hysteresis cycle acts as damping instead.

no $\dot{x}$ in the original sys $\ddot{x} + x = F(\sim) + A\cos\omega t$

$$S(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r\cos\theta, \varepsilon)\sin\theta\, d\theta$$

$$C(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r\cos\theta, \varepsilon)\cos\theta\, d\theta$$

1st segment: $r-2 < x < r \xrightarrow{\text{divided by } r} \psi < \theta < 0$; $\theta = \arccos\left(\frac{x}{r}\right)$, $\psi = \arccos\left(\frac{r-2}{r}\right)$

$$S(r) = \frac{1}{2\pi}\left(\int_0^{\psi} F_I \sin\theta\, d\theta + \int_{\psi}^{\pi} F_{II}\sin\theta\, d\theta + \int_{\pi}^{\pi+\psi} F_{III}\sin\theta\, d\theta + \int_{\pi+\psi}^{2\pi} F_{IV}\sin\theta\, d\theta\right)$$

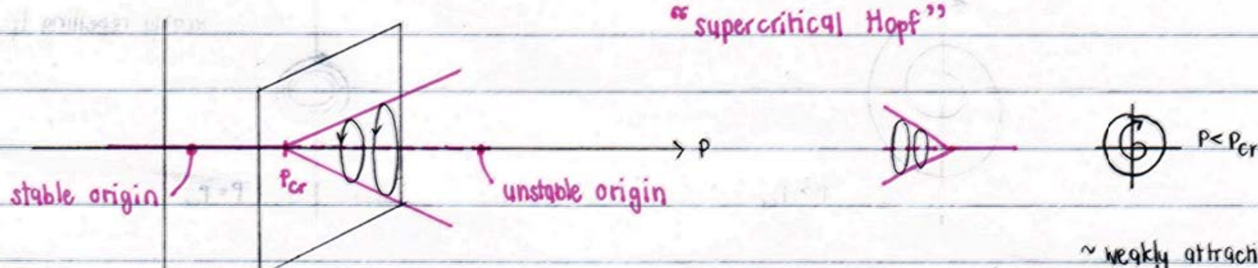
average clockwise, start at $\theta = 0$ from $x = r$ to $x = r-2$, the angle equals to ψ

$$S(r) = \begin{cases} \dfrac{\varepsilon r \sin^2\psi}{\pi}, & r > 1 \\ 0, & r \le 1 \end{cases} \qquad C(r) = \begin{cases} \dfrac{r\varepsilon}{2\pi}\left(\pi - \psi + \frac{1}{2}\sin 2\psi\right), & r > 1 \\ r, & r \le 1 \end{cases}$$

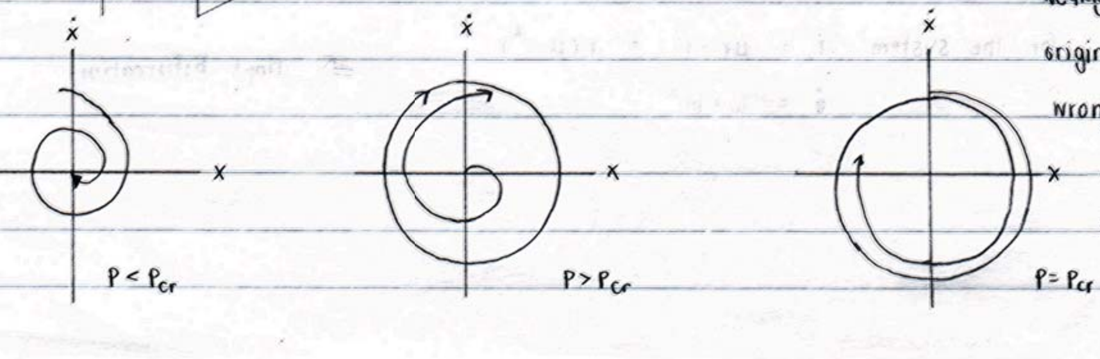
← end of averaging →

October 22, 2009 "Hopf Bifurcation" -- eigenvalues crossing imaginary axis

"supercritical Hopf"

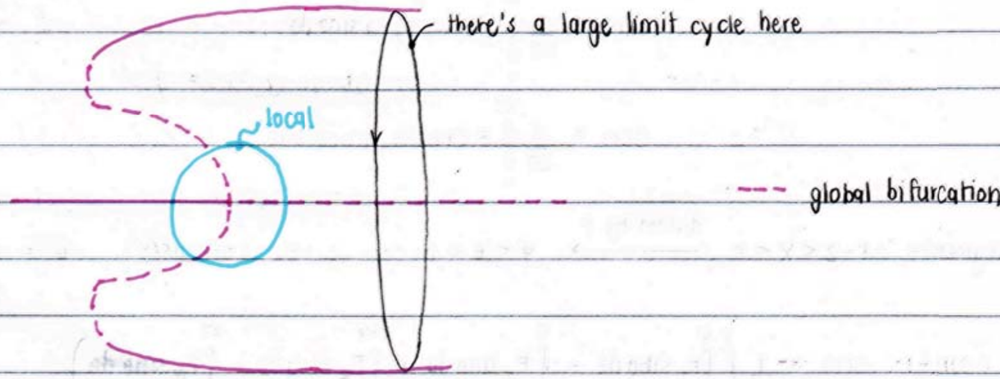


~ weakly attracting origin, can get wrong conclusion to be center instead ~

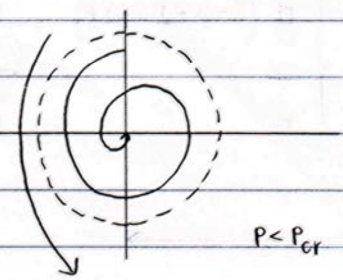


» supercritical Hopf: loss of stability is accompanied by the criterion of stable periodic orbits
↑ soft bifurcation

» subcritical Hopf ← local bifurcation -- there's no other fp showing up in a neighborhood.

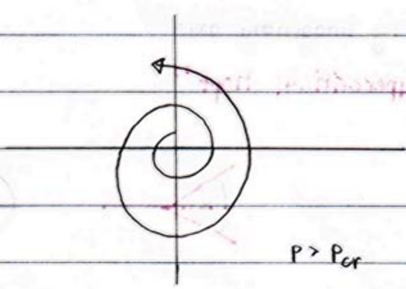


It comes stably .. then BOOM!! becomes unstable -- suddenly blow up
If the amplitude is large enough, you will find large amplitude limit cycle.

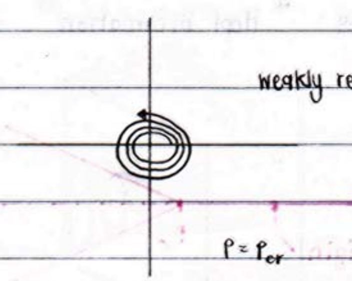


This is a phase portrait of local behavior

$P < P_{cr}$



$P > P_{cr}$



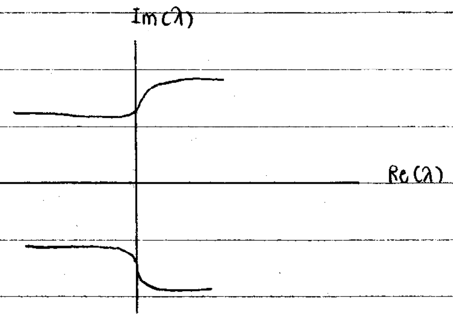
$P = P_{cr}$

for the system $\dot{r} = \mu r - r^3 = r(\mu - r^2)$
$\dot{\theta} = \omega + br^2$
$\Rightarrow$ Hopf Bifurcation

» Hopf Bifurcation Theory: $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, p)$ where $f \in C^+$ (since we have to differentiate to 3rd order)

1) $p = p_{cr}$ $\exists$ $\lambda = \pm i\omega$, all other λ: $Re(\lambda) < 0$

2) transversal root crossing $\frac{d}{dp}(Re\,\lambda)\Big|_{p=p_{cr}} > 0$



crossing but velocity ≠ positive
don't like it., we like it to
cross straight away

usually forget this one

☆☆ 3) at $p = p_{cr}$; $\underset{\sim}{x} = \underset{\sim}{0}$ is weakly attracting/repelling.

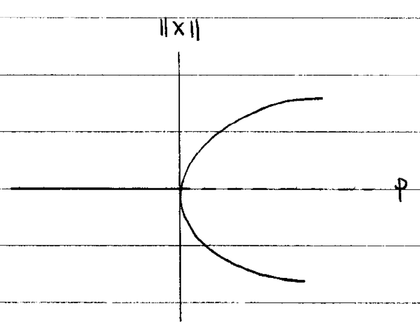
» Normal form for Hopf Bifurcation
$$\dot{x} = -\omega y + f(x,y) \quad ; \; f(\underset{\sim}{0}) = 0 \qquad (*)$$
$$\dot{y} = \omega x + g(x,y) \quad ; \; g(\underset{\sim}{0}) = 0$$

$$J(\underset{\sim}{0}) = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

$$16a = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + \frac{1}{\omega}\Big(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\Big)$$

do Taylor series expansion for 2 variables function

Poincaré - Liapunov constant: $a < 0$ → supercritical Hopf
(for checking conditions #3) $a > 0$ → subcritical Hopf



amp. will scale max as $\|x\| \sim \sqrt{p - p_{cr}}$

Every 2D system that has $\lambda = \pm i\omega$ at $P = P_{cr}$ can be transformed to (*)

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, p) \qquad ; \quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= A(p)\underset{\sim}{x} + \underset{\sim}{g}(\underset{\sim}{x}, p)$$

Introduce change of variables $\underset{\sim}{y} = T\underset{\sim}{x}$

$$\underset{\sim}{x} = T^{-1}\underset{\sim}{y}$$

$$T^{-1}\dot{\underset{\sim}{y}} = A(p)T^{-1}\underset{\sim}{y} + \underset{\sim}{g}(T^{-1}\underset{\sim}{y}, p)$$

$$TT^{-1}\dot{\underset{\sim}{y}} = TAT^{-1}\underset{\sim}{y} + T\underset{\sim}{g}(T^{-1}\underset{\sim}{y}, p)$$

Jordan form -- in our case = $\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$

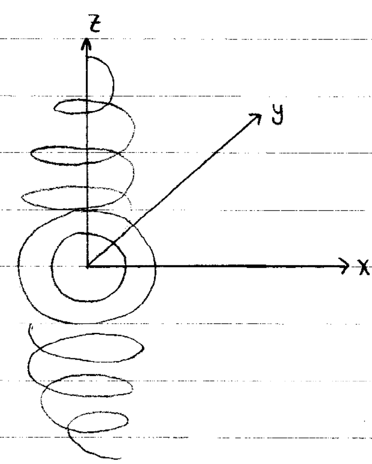
put eigenvalues together

for 2D system!!

got a center

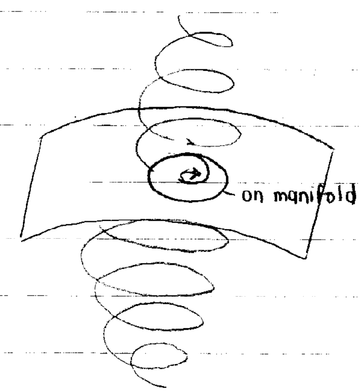
linear system
$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \\ \dot{z} &= -z \end{aligned}$$

decay exponentially



center plane (subspace)

when adding nonlinear: center plane $\longrightarrow$ curve plane = center manifold

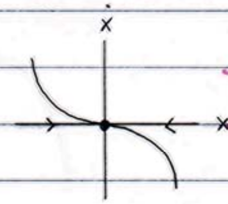


~ center manifold also exist for zero-eigenvalue bifurcation ~

Look at $\dot{x} = x^2 y - x^5$ (*) ☆☆ bonus problem: draw phase portrait for (*) near the origin and see why it's not stable ☆☆

$\dot{y} = -y + x^2$

↑ when $x \approx 0$, $x^2 << y$ then neglect and use $y = 0$ approx. in (*) since it decays to $y = 0$



$\dot{x} = -x^5 \Rightarrow x = 0$ is stable

$\Rightarrow x = y = 0$ is stable

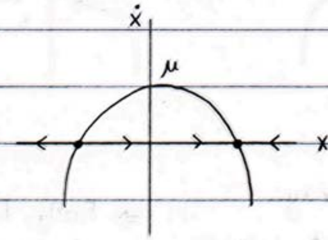
But it's wrong!! The origin is not stable --> failure of the tangent space approximation.

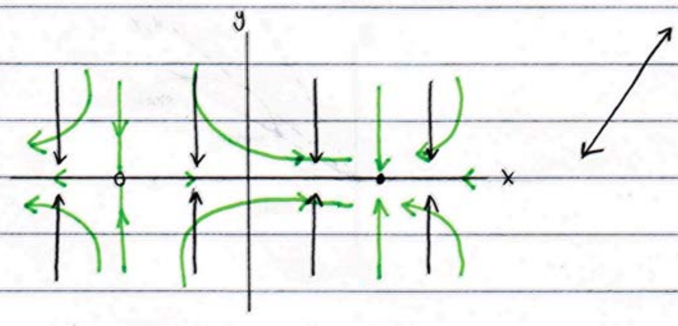
look at linearization; $J = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ --> eigenvalue = 0. then linearization can tell nothing.

October 27, 2009 "Higher Dimension Bifurcations"

» saddle-node $\dot{x} = \mu - x^2 = 0$

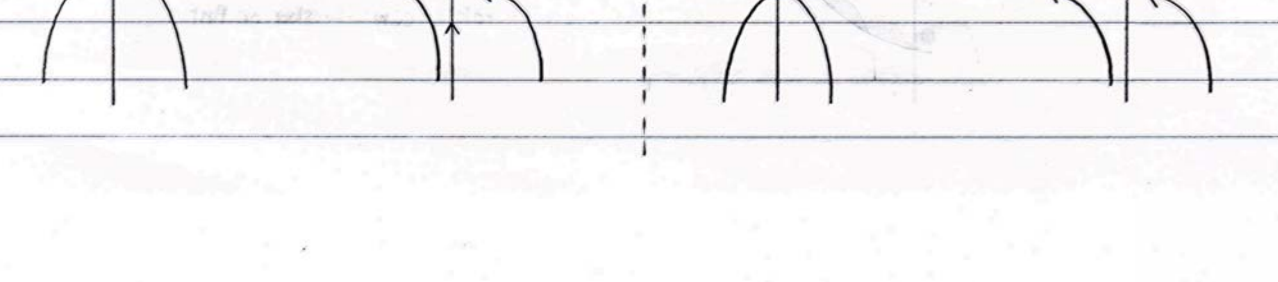
$y = -y = 0$





we know that every flow toward x-axis from nullcline $-y = 0$

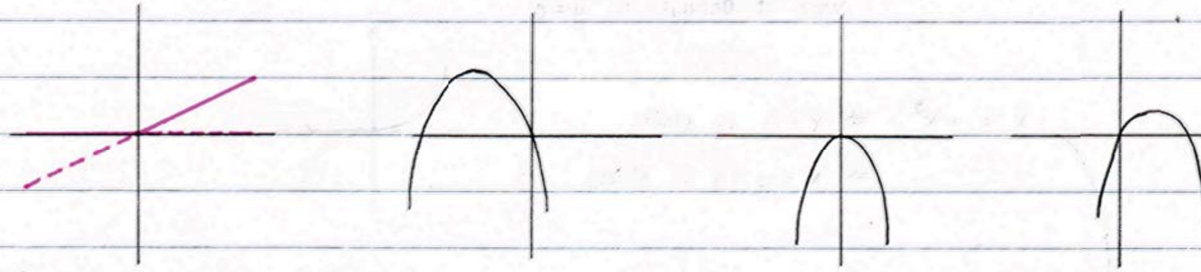
then find out what happened on x-axis



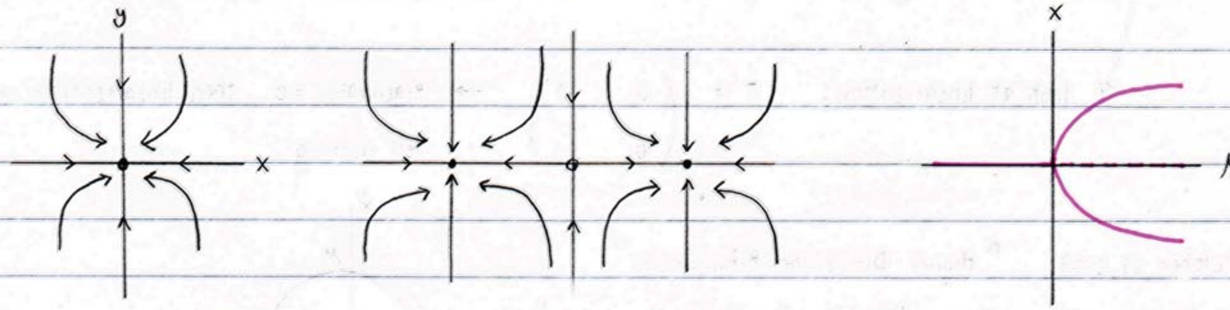
similarly $\dot{x} = \mu x - x^2$ $\quad \dot{y} = -y$ $\quad$ transcritical

$\dot{x} = \mu x - x^3$ $\quad \dot{y} = -y$ $\quad$ supercritical pitchfork

$\dot{x} = \mu x + x^3$ $\quad \dot{y} = -y$ $\quad$ subcritical pitchfork



pitchfork!!



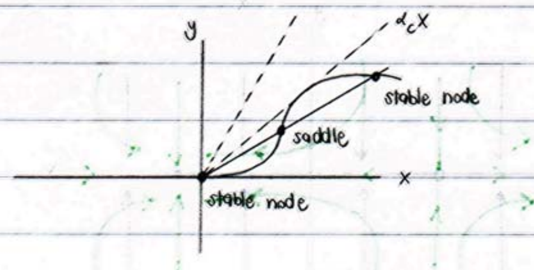
example $\dot{x} = -\alpha x + y$

$\dot{y} = \dfrac{x^2}{1+x^2} - by$

⇒ Firstly, find fp. by drawing nullclines -- intersection = fp

nullclines $y = \alpha x$

$y = \dfrac{x^2}{b(1+x^2)}$



when vary α, saddle-node bifurcation occurs since fp. collapsed.

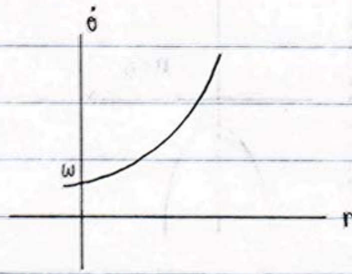
"Global Bifurcation" -- earlier we considered local bifurcation, just small area around fp.

» saddle - node of cycles

$\dot{r} = \mu r + r^3 - r^5 = r(\mu + r^2 - r^4)$ $\qquad \mu < 0, \ \omega > 0, \ b > 0$

$\dot{\theta} = \omega + br^2$ $\qquad$ If $\mu = 0$, we'll get hopf bifurcation.

look at $\dot{\theta}$

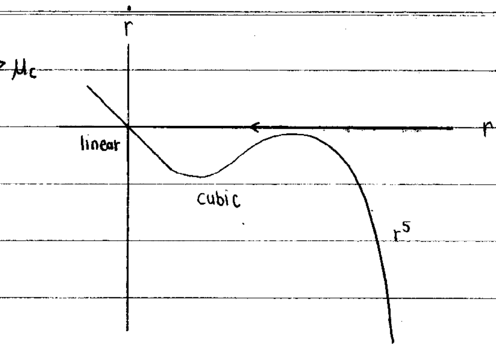


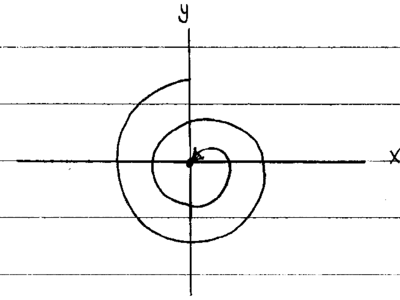
~ nothing's interesting -- no fp.

~ rotate CCW -- slow or fast

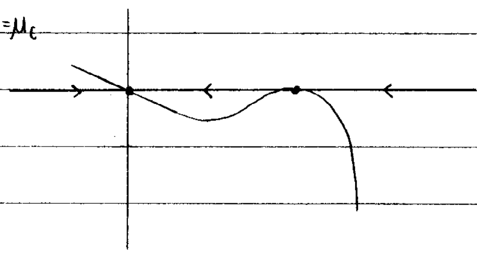
⇒ check $\dot{r}$

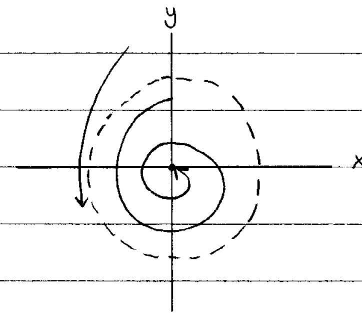
$\mu > \mu_c$



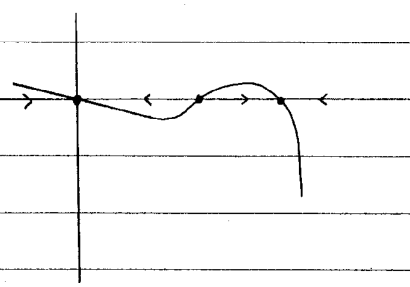


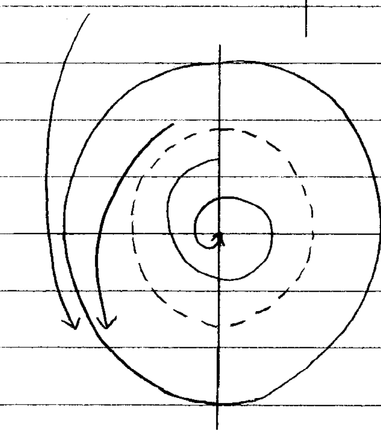
$\mu = \mu_c$





$\mu < \mu_c$

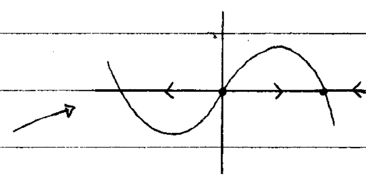




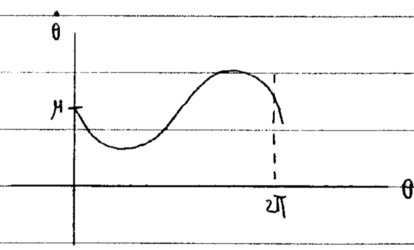
called global bifurcation because what's going on around fp doesn't change
we have to consider large area in order to see that there are limit cycles occur.

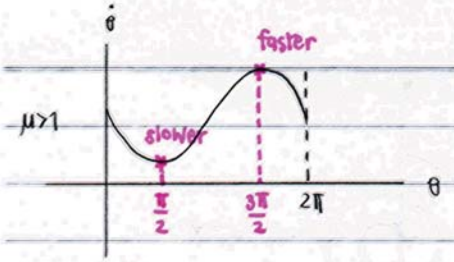
" Infinite Period Bifurcation "

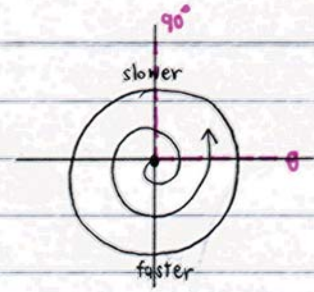
$\dot{r} = r(1-r^2)$ $\dashrightarrow$ fp @ $r = 0, \pm 1$

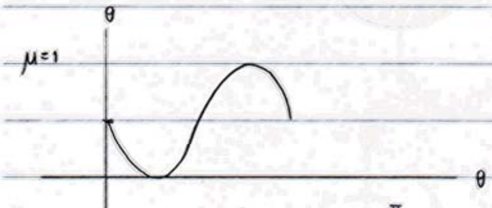


$\dot{\theta} = \mu - \sin\theta$ $\dashrightarrow$ bifurcation in theta direction since parameter is here.

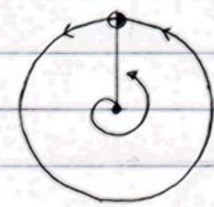




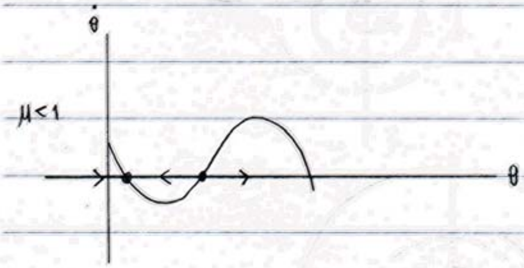


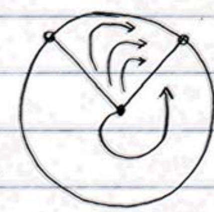


$\dot\theta = 0$ when $\theta = \frac{\pi}{2}$



the spiral can't cross the line and when on the cycle, it takes infinite to come back





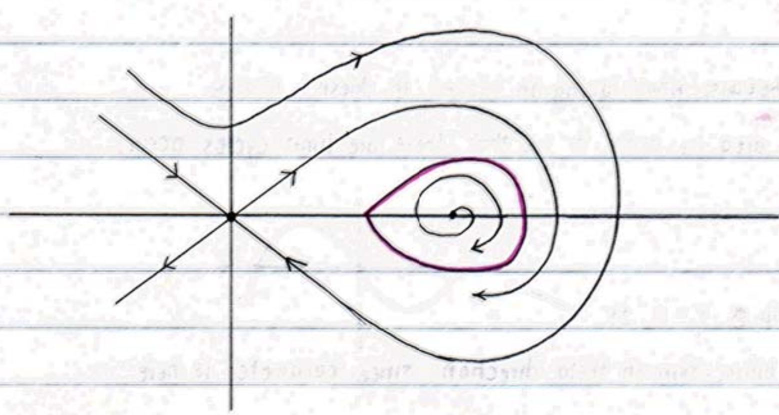
"Homoclinic Bifurcation"

$$\dot x = y$$
$$\dot y = \mu y + x - x^2 + xy$$

$\Rightarrow$ nullclines to get fp

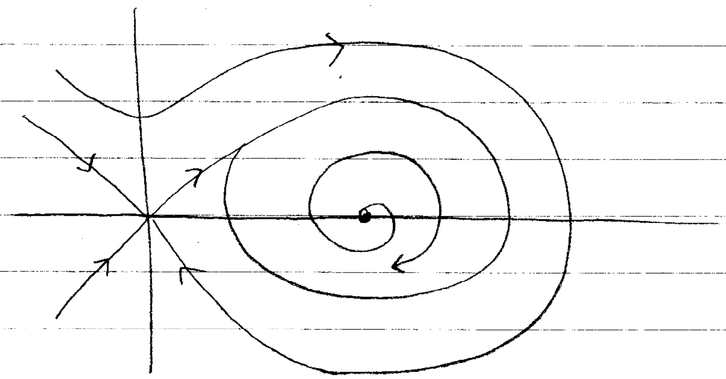
when $y=0$, left only $x(1-x)=0$

then $x = 0,1$

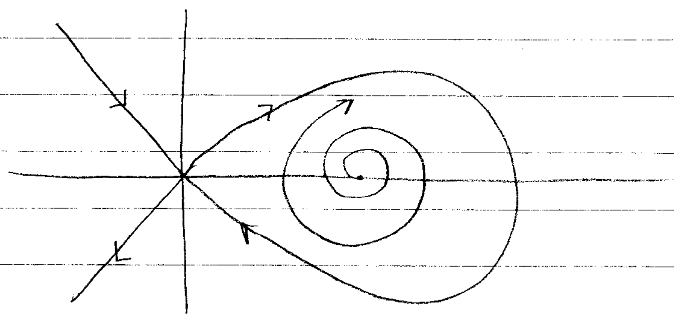


trajectory can't across the manifold

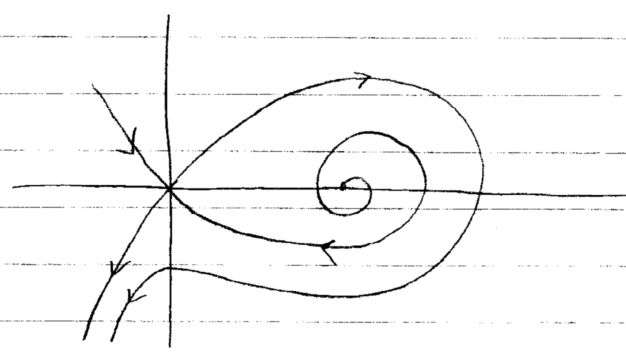
when $\mu \to \mu_c$, expand limit cycle



$\mu = \mu_c$ connect!!



$\mu > \mu_c$ connection break into another direction.



October 29, 2009 "Center Manifold Approximation"

↳ attracting manifold -- what happen on center manifold can explain local behavior of sys.

$\dot{y} = Ay + F_2(y) + F_3(y) + \mathcal{O}(\|y\|^4)$; $\underset{\sim}{y} \in \mathbb{R}^n$

where $\lambda_1, \dots, \lambda_m$ = eigenvalues w/ $Re(\lambda_i) = 0$

$\lambda_{m+1}, \dots, \lambda_n$ = —— " —— $Re(\lambda_i) < 0$

$P_1, \dots, P_m, P_{m+1}, \dots, P_n$ = generalized eigenvectors corresponding to eigenvalues

After Jordan transformation $\underset{\sim}{y} = P\underset{\sim}{v}$ where $P = [P_1, \dots, P_n]$

$$\dot{\underset{\sim}{v}} = J\underset{\sim}{v} + P^{-1}F_2(P\underset{\sim}{v}) + P^{-1}F_3F(P\underset{\sim}{v}) + \dots$$

where $J = P^{-1}AP = \begin{bmatrix} J_{c\,(m\times m)} & 0 \\ 0 & J_{s\,(n-m)\times(n-m)} \end{bmatrix}$ J_c = Jacobian center, J_s = Jacobian stable

Therefore, divide into center and stable

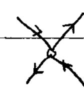
$$\dot{\underset{\sim}{v}}_c = J_c\underset{\sim}{v}_c + \underset{\sim}{G}_2(\underset{\sim}{v}_c, \underset{\sim}{v}_s) + \underset{\sim}{G}_3(\underset{\sim}{v}_c, \underset{\sim}{v}_s) \quad (*)$$

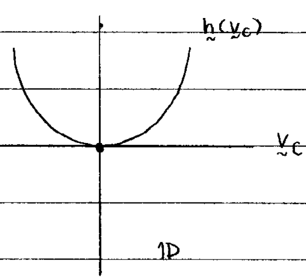
$$\dot{\underset{\sim}{v}}_s = J_s\underset{\sim}{v}_s + \underset{\sim}{H}_2(\underset{\sim}{v}_c, \underset{\sim}{v}_s) + \underset{\sim}{H}_3(\underset{\sim}{v}_c, \underset{\sim}{v}_s)$$

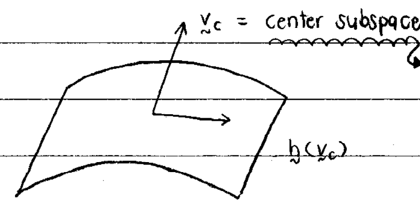
local center manifold -- it's a graph $\underset{\sim}{v}_s = \underset{\sim}{h}(\underset{\sim}{v}_c)$

Carr, 1981

~ local invariant, not like







$\underset{\sim}{h}(\underset{\sim}{0}) = \underset{\sim}{0}$, $D_{\underset{\sim}{v}_c}\underset{\sim}{h}_i(\underset{\sim}{0}) = \underset{\sim}{0}$ tangent to the center subspace

plug $\underset{\sim}{v}_s = \underset{\sim}{h}(\underset{\sim}{v}_c)$ into (*) and get

$$D_{\underset{\sim}{v}_c}\underset{\sim}{h}(\underset{\sim}{v}_c)\left(J_c\underset{\sim}{v}_c + \underset{\sim}{G}_2(\underset{\sim}{v}_c, \underset{\sim}{h}(\underset{\sim}{v}_c)) + \underset{\sim}{G}_3(\underset{\sim}{v}_c, \underset{\sim}{h}(\underset{\sim}{v}_c)) + \dots\right) = J_s\underset{\sim}{h}(\underset{\sim}{v}_c) + \underset{\sim}{H}_2(\underset{\sim}{v}_c, \underset{\sim}{h}(\underset{\sim}{v}_c)) + \underset{\sim}{H}_3(\underset{\sim}{v}_c, \underset{\sim}{h}(\underset{\sim}{v}_c)) + \dots$$

next step is set $\underset{\sim}{h}$ as a polynomial and equate coeffs.

↳ when $\underset{\sim}{h}$ is known, then we know every term in (*)

stability of (*) same as the original sys.

» h as a polynomial: $\underset{\sim}{h} = \begin{pmatrix} h_{11}x^2 + h_{12}xy + h_{13}y^2 \\ h_{21}x^2 + h_{22}xy + h_{23}y^2 \end{pmatrix}$ ⇝ example in 2D

equates coeffs. to find $h_{11}, h_{12}, \dots$ Then eg. use Bantin Formula for a Hopf bifurcation.

example $\dot{x} = x^2y - x^5$ ☆☆ bonus problem: do this with $\dot{x} = x^2y - x^3$ ☆☆

$\dot{y} = -y + x^2$

$\exists$ $y = h(x)$ local center manifold -- $h(x) = ax^2 + bx^3 + \mathcal{O}(x^4)$

$\mathcal{N}(h(x)) = Dh(x)(Ax + f(x, h(x))) - Bh(x) - g(x, h(x)) = 0$

assume that we have 2D system of this form

$\begin{bmatrix} \dot{x} = Ax + f(x,y) \\ \dot{y} = By + g(x,y) \end{bmatrix} \Rightarrow \begin{matrix} A = 0, \ B = -1 \\ f = x^2y - x^5, \quad g = x^2 \end{matrix}$

$\mathcal{N}(h(x)) = (2ax + 3bx^2 + \dots)(ax^4 + bx^5 - x^5 + \dots) + ax^2 + bx^3 - x^2 + \dots = 0$

equate coeffs x^2: $a - 1 = 0$ -- $a = 1$

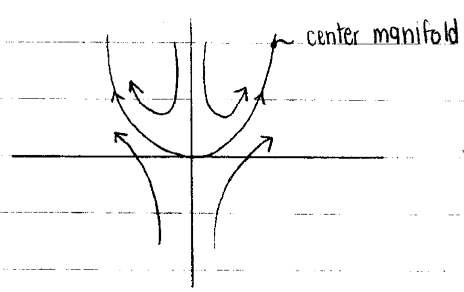
x^3: $b = 0$

$\Rightarrow y = x^2 + \mathcal{O}(x^4)$

got a center manifold!!

then go back to $\dot{x} = Ax + f(x, h(x))$

$\dot{x} = Ax + f(x, h(x)) \Rightarrow \dot{x} = x^4 + \mathcal{O}(x^5)$ -- the stability of the origin is determined from this on the center manifold.. unstable on the center manifold but manifold itself is stable



phase portrait of the original system

~ In higher dimension, center manifold can be painful, usually used for checking answers.

Δ is a product of eigenvalue , τ is a sum of eigenvalue

" Driven Pendulum" — constant driving eg. constant torque

$\phi'' + \alpha\phi' + \sin\phi = I$; $I > 0$, $\alpha > 0$

write in phase space form

$\phi' = y$ -- live on circle

$y' = I - \sin\phi - \alpha y$ -- real number

Thus, phase space will be infinite cylinder

↳ no need to.. but should make an understanding on a correct phase space.

fixed point: $y = 0$

$I - \sin\phi - \alpha y = 0 \dashrightarrow I = \sin\phi$

↳ how many fp? -- depends on I
- $I < 1$ ~ 2 fp
- $I > 1$ ~ no fp. , since $|\sin\phi| < 1$

$$A = \begin{pmatrix} 0 & 1 \\ -\cos\phi & -\alpha \end{pmatrix} \qquad \left.\begin{array}{l} \tau = -\alpha < 0 \\ \Delta = \cos\phi = \pm\sqrt{1-I^2} \end{array}\right\} \text{saddle / sink}$$

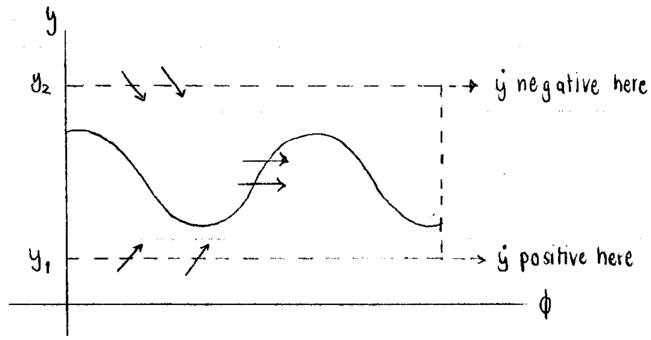
$\Delta > 0$ if $\tau^2 - 4\Delta > 0 \dashrightarrow$ stable node

$I = 1$ -- saddle-node bifurcation

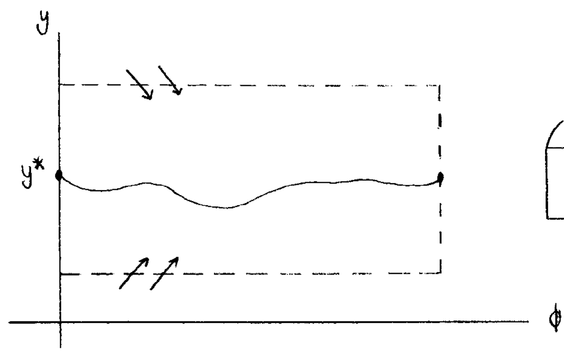
Then periodic orbits!!

↳ $I > 1$ no fp. but can still see periodic orbits

conjunction: $\exists$ stable limit cycle (closed orbit on cylinder but not close on x-y)

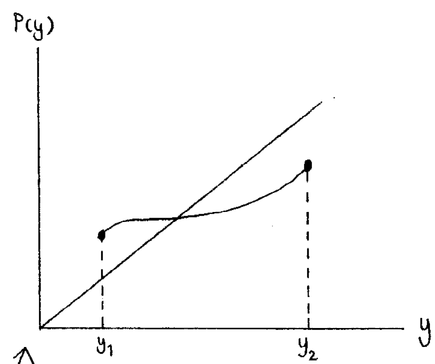


consider $y = \frac{1}{\alpha}(I - \sin\phi)$, $y' = 0$



Poincaré Map (first return map)

$\boxed{P(y^*) = y^*}$ There's periodic orbit if function like this can be found.



if function intersect this graph, then there is fp but we can't tell what analytical solution is.

P(y) is continuous, no need to be smooth

P(y) is monotonic because of uniqueness.

we know from this that we have fp

That means we have closed orbit, also unique fp.

Nov 3, 2009 $\left\langle \frac{dr}{dt} \right\rangle \stackrel{?}{=} \frac{d\langle r\rangle}{dt}$ -- How do I do this?

book: Feynman

diff. under the integral signs

motivating idea: discrete

$$a_i = \frac{dr}{dt_i} \qquad i = 1, \dots, N$$

$$\langle a_i \rangle = \frac{1}{N}\Sigma a_i = \Sigma \frac{1}{N} a_i$$

$\langle\ \rangle \int$ $\qquad \int \langle\ \rangle$ -- can commute

what about $\langle r \rangle$? $\qquad \langle r\rangle = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} r(T)\,dT \longleftrightarrow \frac{1}{2\pi}\int_{0}^{2\pi} r(T)\,dT$

boundary change w/ time -- problem!!

but $x = r(t)\cos(t + \phi(t))$

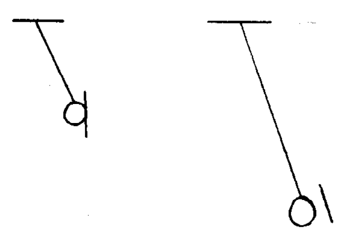
is a slowly varying periodic = periodic in slow time

can shift anywhere then $t-\pi \to t+\pi$ can become $0 \to 2\pi$

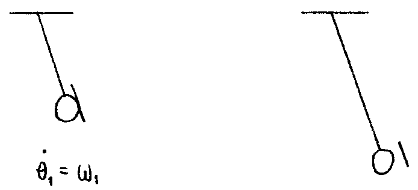
"Coupled Oscillators" -- think about blinker in our car and car in the front

sometimes, they seem to blink together and overlap later.

$\hookrightarrow$ model it as a pendulum



mass has no effect to period, only length blink when touch the pad



$\dot{\theta}_1 = \omega_1$ $\qquad$ $\dot{\theta}_2 = \omega_2$

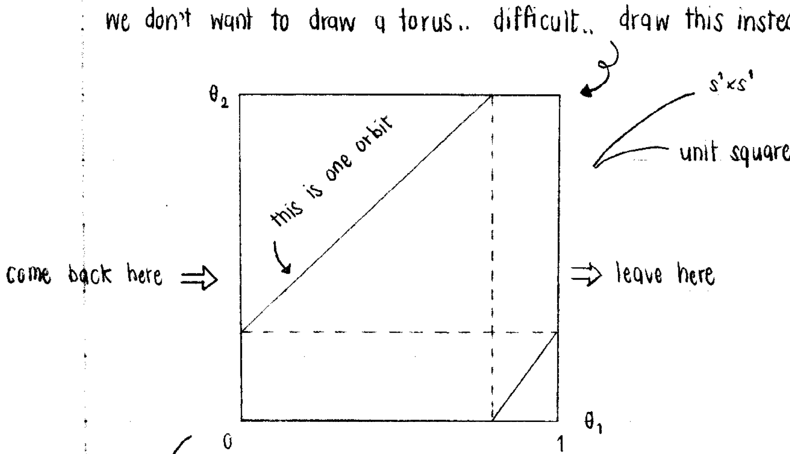
Now, there are 2 uncoupled system

interested in system $\begin{bmatrix} \dot{\theta}_1 = \omega_1 \\ \dot{\theta}_2 = \omega_2 \end{bmatrix}$

correct phase space = torus -- we have θ_1, θ_2 two angular variables (toroidal phase space)

$S^1 \times S^1$

we don't want to draw a torus.. difficult.. draw this instead



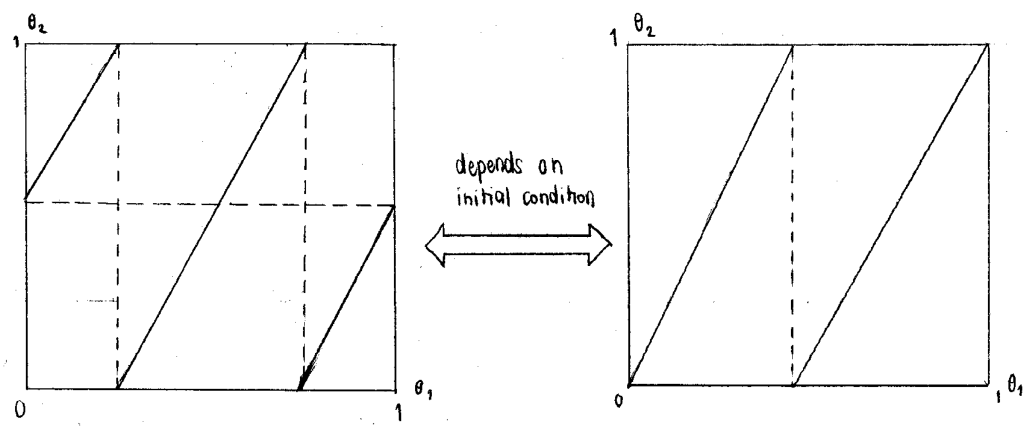
$$\frac{d\theta_1}{dt} = \dot{\theta}_1 = \omega_1$$

$$\frac{d\theta_2}{dt} = \dot{\theta}_2 = \omega_2$$

$$\Rightarrow \frac{d\theta_2}{d\theta_1} = \frac{\omega_2}{\omega_1}$$

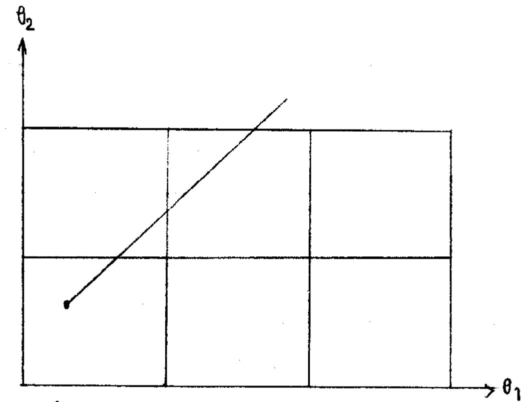
got periodic orbits on torus -- don't say limit cycles because this's still linear

unit square



if it's not a unit,

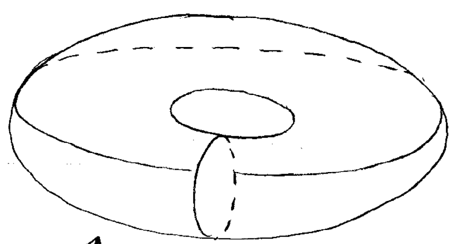
but it's real



make 1.1 become 0.1

by modulus function

many unit squares.. put other squares on the first one.



rational number (Q)

$$\omega_2 = \frac{m}{n}\,\omega_1 \Rightarrow \text{closed knots}$$

there're m trajectories wrap around

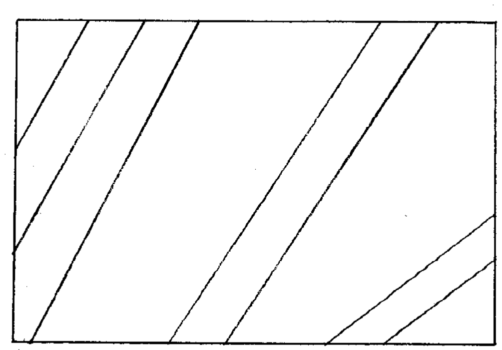
— " — n ——— " ———

no fp in this torus, trajectory can't cross

2 surfaces but 3D object

Q: $\omega_2 = \alpha\omega_1$ if $\alpha \in Q^*$ (irrational)

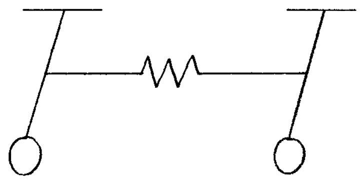
A: don't get closed orbit.. densely fills the torus



almost

quasi-periodic trajectory

uncoupled.. will add coupling



$$\dot{\theta}_1 = f_1(\theta_1, \theta_2)$$
$$\dot{\theta}_2 = f_2(\theta_1, \theta_2)$$
--> f_1, f_2 are periodic

phase difference --> if equal zero, we'll get synchronize

$$\dot{\theta}_1 = \omega_1 + k_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + k_2 \sin(\theta_1 - \theta_2)$$

$k_1, k_2 > 0$ can write as

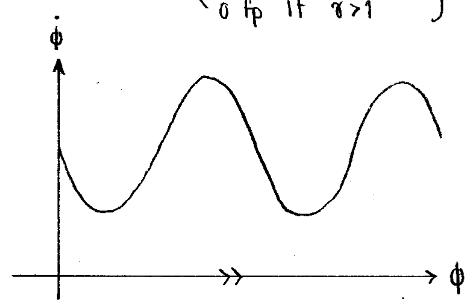
$$\dot{\theta}_1 = \omega_1 - k_1 \sin(\theta_1 - \theta_2)$$
$$\dot{\theta}_2 = \omega_2 + k_2 \sin(\theta_1 - \theta_2)$$

phase difference: $\phi = \theta_1 - \theta_2$

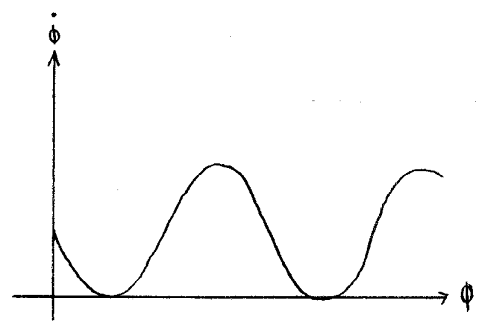
$\dot{\phi} = \omega_1 - \omega_2 - (k_1 + k_2)\sin\phi$ --> one dimensional system --> flow on a circle

fp: $\gamma = \dfrac{\omega_1 - \omega_2}{k_1 + k_2} = \sin\phi$
- 2 fp if $\gamma < 1$
- 1 fp if $\gamma = 1$
- 0 fp if $\gamma > 1$

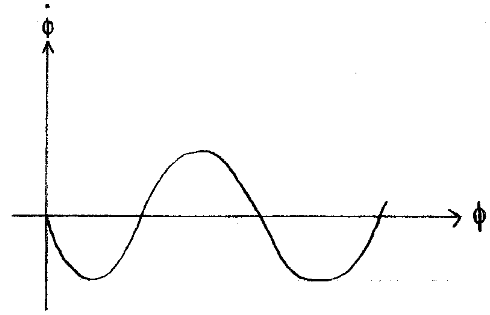
saddle-node bifurcation



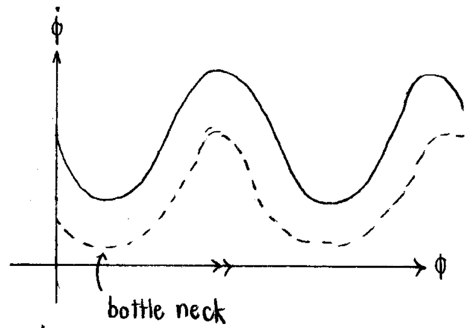
$|\gamma| > 1$ -- no synchronization



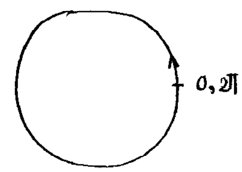
$|\gamma| = 1$



$|\gamma| < 1$



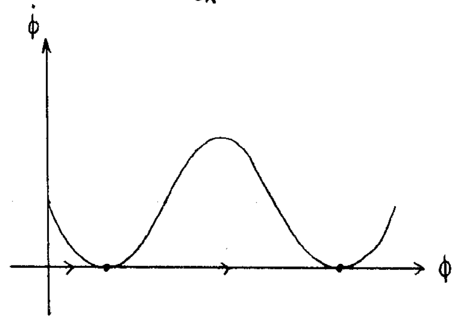
flow go in one direction

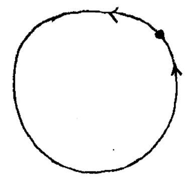


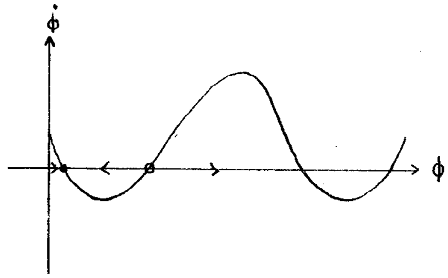
circle

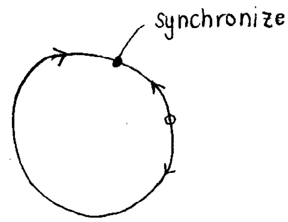
$\phi \in S^1$

phase different
เพิ่มขึ้นเรื่อยๆ

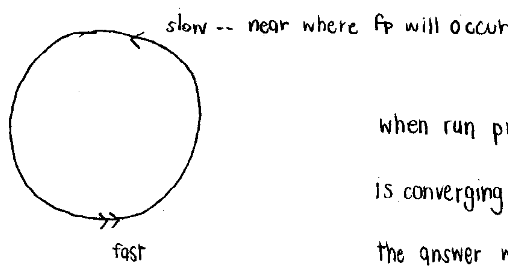








bottle neck case:



when run program, we'll see that the answer is converging to fp but if run a little longer, the answer will run through bottle neck.

asymptotic behavior summary: 1D -- fixed point (only)

2D -- limit cycle, fp, cycle graph

"3D Behavior: Chaos"

Lorenz system

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

need to check about conservative .. if trajectory is coming back

$\sigma, r, b > 0$ --> Rayleigh - Bernard convection waterwheel

if there's only $-bz$

everything will collapse to 2D

Symmetry $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$

1) find f.p. $(0,0,0)$

always fp. -- there's a bifurcation change its stability ex. pitchfork

$y = x$: $\quad x^2 - bz = 0$

$x[(r-1) - z] = 0 \Rightarrow z = r-1 \rightarrow x = \pm\sqrt{b(r-1)}$

exist only if $r > 1$

$r \geq 1 \quad C^+ = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$

$C^- = (-\sqrt{b(r-1)}, \sqrt{-b(r-1)}, r-1)$ (fp = rotation of waterwheel)

there's a bifurcation at $r=1$ -- pitchfork since there's only $(0,0,0)$ at first and then become 3 pt.

2) check stability to see if it's super/subcritical pitchfork bifurcation

3) can the system be conservative?

- adding the flow into the system -- sink / source (fp)

  can't have in conservative sys.

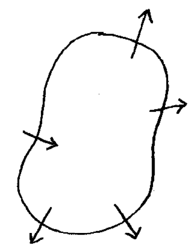
- center, saddle - point

GAS = globally asymptotically stable

$$\dot{V} = \int \underset{\sim}{f} \cdot \underset{\sim}{n} \, dA$$

RHS of the system

$$= \int \nabla \cdot \underset{\sim}{f} \, dv \quad \text{-- easier}$$



~ For dissipative sys, any volume in phase space will shrink as time goes by

↳ volume → 0 but not fp.

$$\underset{\sim}{f} = \begin{pmatrix} \sigma(y-x) \\ rx - y - xz \\ xy - bz \end{pmatrix}, \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$\nabla \cdot \underset{\sim}{f} = -\sigma - 1 - b < 0$ -- Thus, not conservative. it's dissipative

$$V = V_0 e^{(-\sigma-1-b)t}$$

Nov 10, 2009 "Lorenz's system, Chaos"

↳ got new phenomenon (not fp, limit cycle, quasi-periodic anymore)

$$\left.\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}\right\} \quad \begin{aligned} &\text{fp exists } (0,0,0) \\ &C^+, C^- = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1) \sim r > 1 \end{aligned}$$

↳ $r=1$ pitchfork bifurcation

Lorenz's system

1) symmetry

2) volume contraction -- in this case, not fp, but a strange attractor

↳ if put volume in the system that has stable fp.

then flow act on volume, make volume shrink

3) no quasi - periodic solutions

↳ happen on torus = invariant volume -- contradict to volume contraction.

4) no repelling fp. or repelling limit cycle --→ volume increase

↳ but ok with sink or saddle

> check linear stability of the origin

neglect nonlinear term

$$\dot{x} = \sigma(y-x)$$

$$\dot{y} = rx-y$$

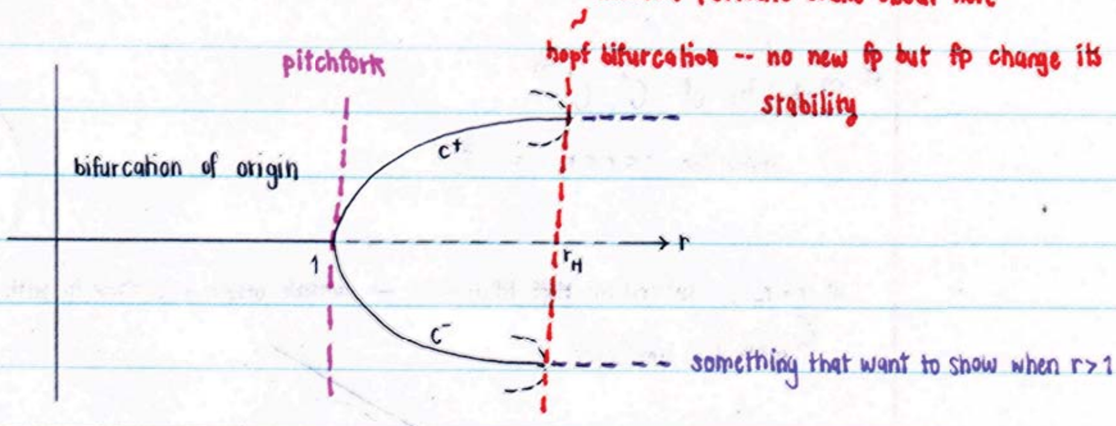
$\dot{z} = -bz$ -- contracting and decouple $z \to 0$

from $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \dashrightarrow$ $\tau = -\sigma - 1 < 0$

$\Delta = \sigma(1-r)$ ☆☆ depends on this one

$r>1$ -- saddle

$r<1$ -- sink $\tau^2 - 4\Delta = (\sigma+1)^2 - 4\sigma(1-r) = (\sigma+1)^2 + 4\sigma r > 0$ -- stable node unstable periodic orbits occur here

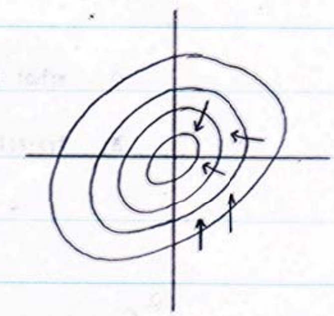


r<1: origin is globally asymtotically stable (GAS)

↳ no matter where the sys starts, it will end up at the origin.

something interesting happens when r>1.

"Lyapunov Function, V"



v>0 positive definition -- elliptical type of obj.

v<0 negative definition -- flow going into the obj.

$v = \frac{1}{\sigma}x^2 + y^2 + z^2$ -- usually starts with square terms to guarantee positive definition

but $\sin x + 2$ also give positive definition

$\dot{v} = \frac{1}{\sigma}2x\dot{x} + 2y\dot{y} + 2z\dot{z} - 2\left[(r+1)xy - x^2 - y^2 - bz^2\right]$

$= 2\left[-\left(x - \frac{r+1}{2}y\right)^2 - \left(1 - \left(\frac{r+1}{2}\right)^2\right)y^2 - bz^2\right] < 0$ considering $r>1$

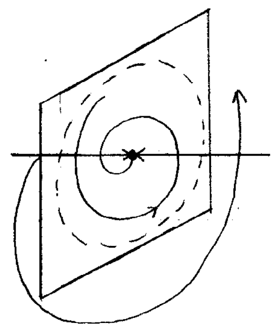
"Stability of C^+, C^-"

stable for $1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$

not really repelling limit cycle. There's a stable direction

at $r = r_H$; subcritical Hopf bifurcation $\rightarrow$ unstable origin + 2 stable fp with unstable limit cycle around

saddle type limit cycle



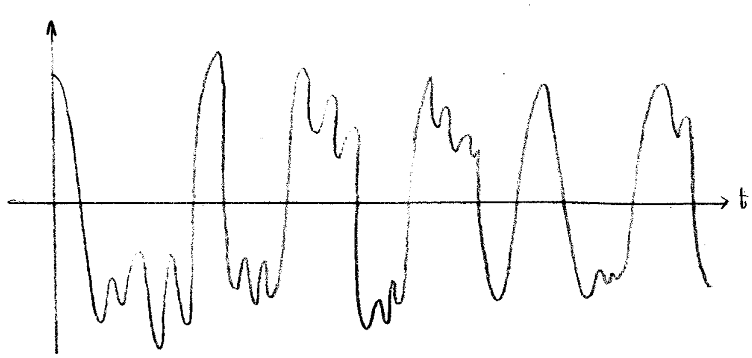
Q: what happens when $r > r_H$?

A: trajectories enter a large set.. behavior is bounded $\Rightarrow$ $\boxed{\text{solutions go to "strange attractor"}}$

- stable limit cycle? no!!

"Chaos on the Strange Attractor"

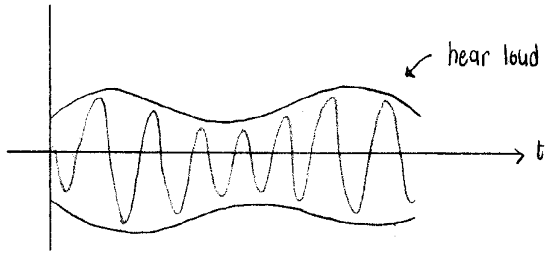
$\sigma = 10$, $b = \frac{8}{3}$, $r = 28$, $r_H = 24.74$



chaos definition: long-term aperiodic behavior

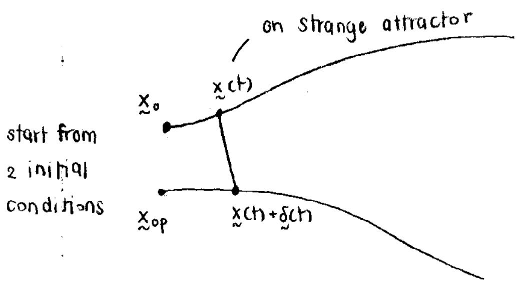
sensitive dependence on lil initial condition in a deterministic system.

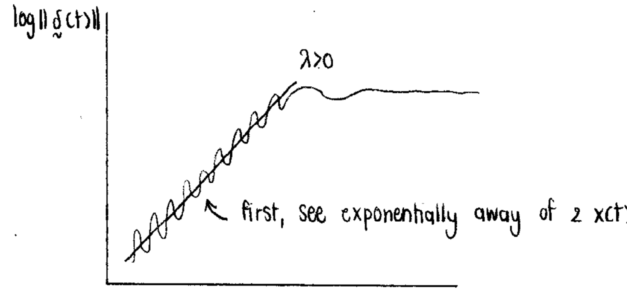
- beat harmonic -- combine 2 periodic solution with close frequencies and get time history below



largest eigenvalue - but not eigenvalue of phase space

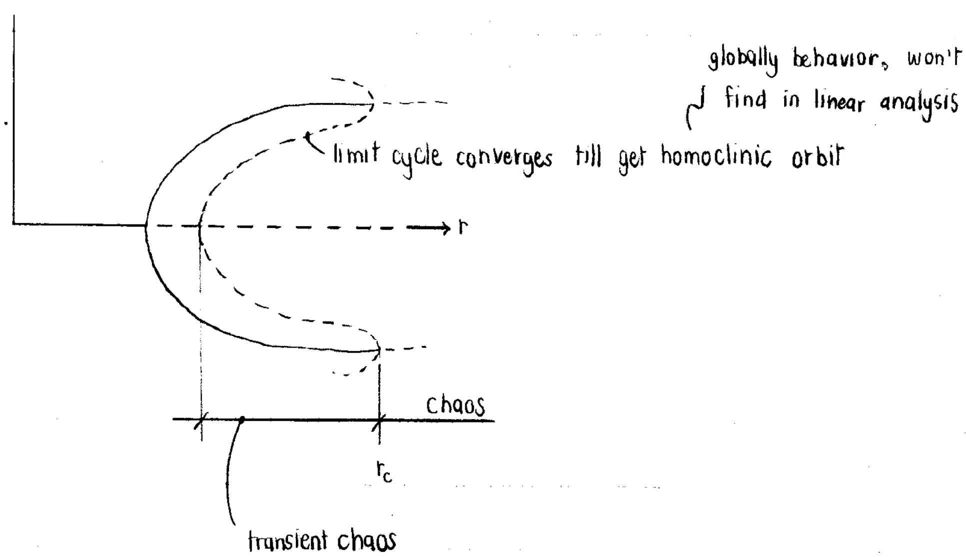
$\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$ Lyapunov exponent

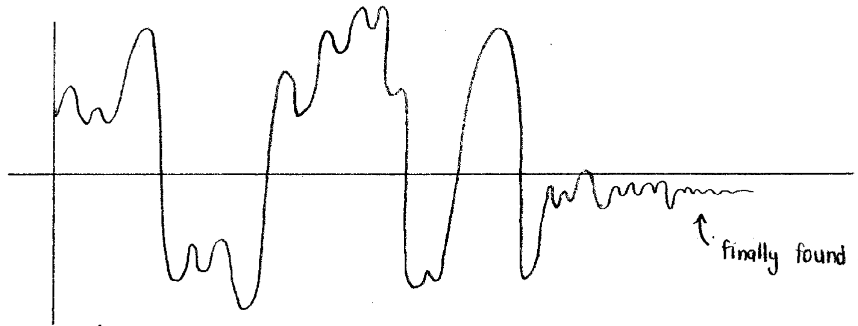




At color page 2: At first, red points are close and then go away from each other but still separate while on strange attractor $\longrightarrow$ butterfly

Nov 12, 2009 • volume = 0 in 3D --> points, lines, planes





finally found stable pt.

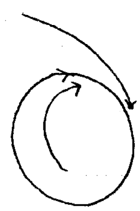
seem to be chaos.. but no. only that period of transient chaos is quite long

$\lim_{r \to r_c} t_{\text{transient}} \to \infty$

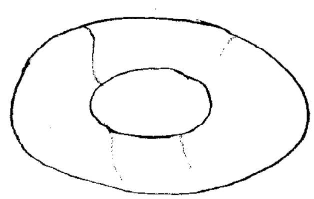
- Lorenz attractor -- the butterfly one. looks like two 2D planes link together but it's not like that. There's a little thickness to give some room for trajectories not cross each other. becomes ≈ 2.05 dimensional

<u>Attractor</u> - sink

- stable limit cycle



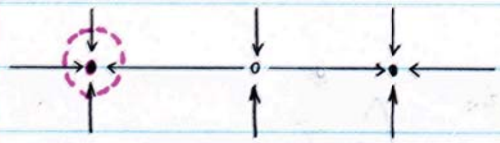
- ...sth.. on torus



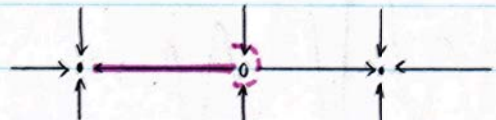
what are common properties in these 3?

talking about the set.

1) "attracts" for an open neighborhood of object.
2) the attractor is invariant (start on it, stay on it)

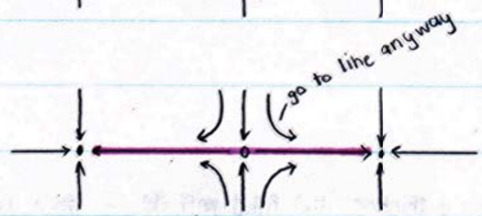


satisfy 2 conditions



invariant.. start on line, stay there

but for open set --- it doesn't attract



satisfy 2 conditions

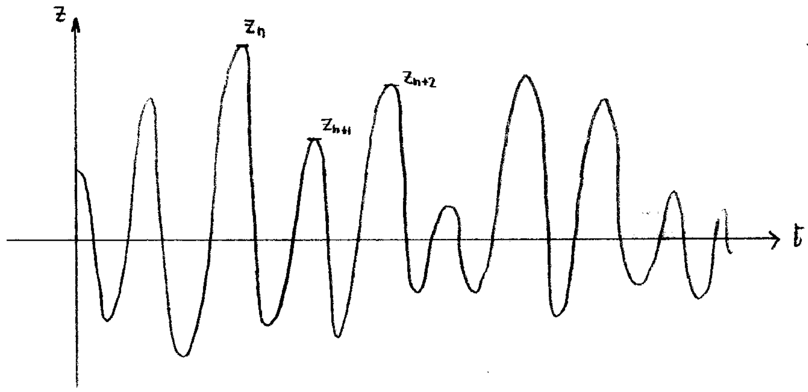
???

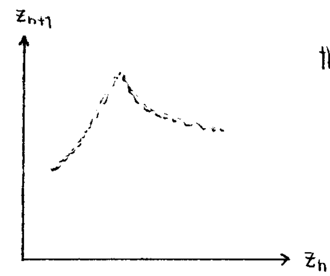
properties (cont.) 3) minimality $A \subseteq B$

↳ an attractor -- no proper subset $A \subset B$

» When having several attractors, hysteresis may occur «

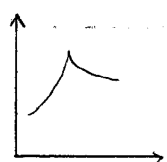
- Lorenz attractor = strange attractor: sensitive dependence on initial condition.



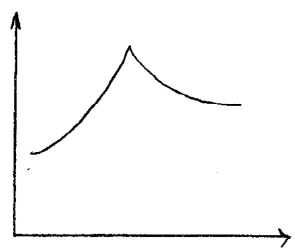


there's a thickness that filled with dot -- then Lorenz said

why don't we find a smooth function to predict z_{n+1} from z_n?



↳ would correspond to periodic orbit



$z_{n+1} = f(z_n)$ -- map

$z^* = f(z^*)$ -- periodic orbit on Lorenz system

Nov 24, 2009

» What is superstable 2-cycle? $\leftrightarrow$ what is superstable fixed point?

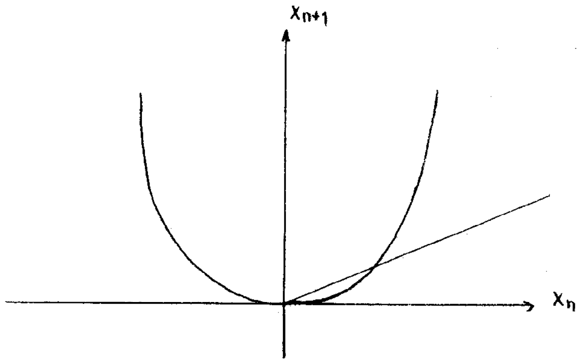
For map, fixed pt. is $x^* = f(x^*)$

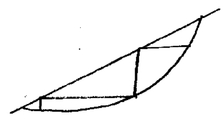
$$\left.\begin{array}{l} |f'(x^*)| < 1 \\ |f'(x^*)| > 1 \end{array}\right\}$$ linearized eq$^{\underline{n}}$ $x_{n+1} = f'(x^*) x_n$

$x_{n+1} = a^n x_0$

what happened if $a = 0$?

$x_{n+1} = x_n^2$



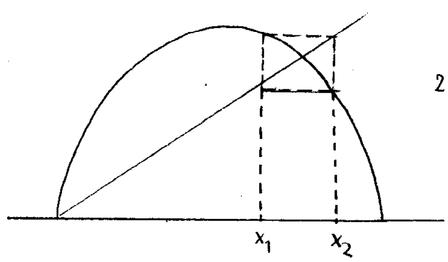


superstable!!
(converge very fast)

» stability of period-n orbits

let's see $n=2$ $\quad f(f(x_1)) = x_1$ -- we got period-2 orbits from this map.

To see its stability, perturb x and see the rate of that perturbation



2 intersections $\dashrightarrow$ period-2 orbits

$f(x_1) = x_2$, $f(x_2) = x_1$

By linearization, $\delta_{n+1} = f'(f(x_1)) f'(x_1) \delta_n$

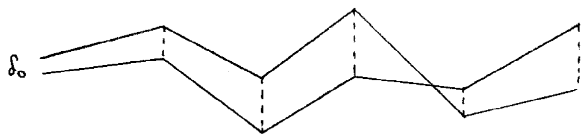
$= f'(x_2) f'(x_1) \delta_n$

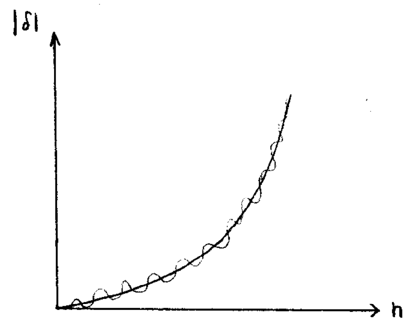
$\delta_{n+1} = \left(\prod_{i=1}^{n} f'(x_i) \right) \delta_n$

$\hookrightarrow |\prod f'(x_i)| < 1$ -- orbit is stable superstable period-2 orbit

$$f'(x_1)f'(x_2) = 0 \dashrightarrow f'(x_1) = 0 \text{ or } f'(x_2) = 0$$

sensitive dependence on initial conditions $\longleftrightarrow$ Lyapunov exponent





after some time, it will saturate

$$\|\delta_n\| \sim \prod f'(x_i)\|\delta_0\| \longleftrightarrow \|\delta_n\| \sim \lambda^n\|\delta_0\|$$

equivalent growth rate -- think as Lyapunov exponent

log of product = sum of log!!

$$\lambda^n = \prod f'(x_i)$$

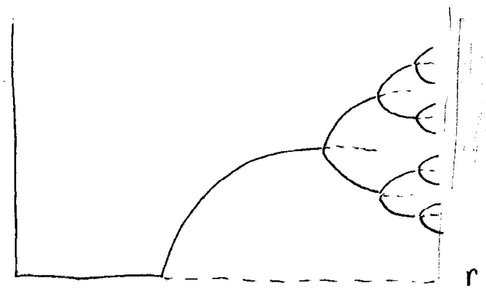
$$n\log\lambda = \sum \log f'(x_i) \Rightarrow \log\lambda = \frac{1}{n}\sum\log f'(x_i) \Rightarrow \lambda = e^{\frac{1}{n}\sum\log f'(x_i)}$$

Lyapunov exponent -- volume contraction for the continuous system (trajectory)

Flogret exponent -- volume contraction for a map.

Logistic map & Tent map

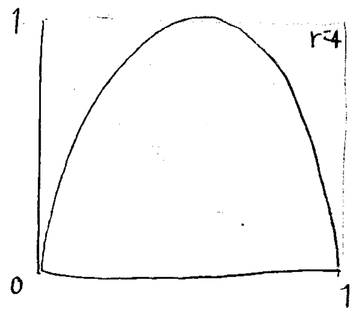
consider when $r=4$ -- $x_{n+1} = 4x_n(1-x_n)$

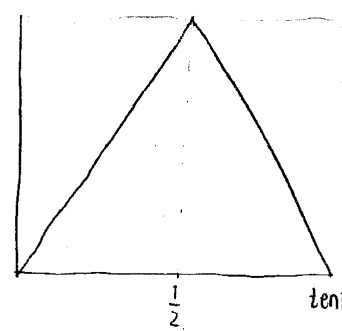


chaos but there're periodic windows if we're looking for them.

when $r=4$ -- cover full map

$$\begin{cases} 2x, & x < \frac{1}{2} \\ 2-2x, & x \geq \frac{1}{2} \end{cases}$$





tent map!?

$$f_{n+1} = f_n(x) \xrightarrow{\text{homeomorphism}} g_{n+1} = g_n(x)$$

$\hookrightarrow g = h \circ f \circ h^{-1}$

$g \circ h = h \circ f$

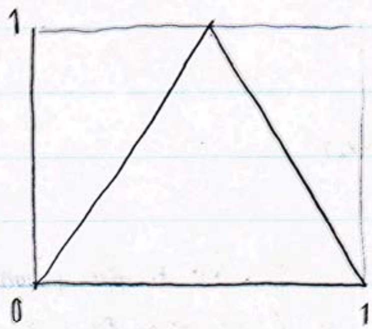
change of variables: $x_n = \sin^2\theta_n$

$$\sin^2\theta_{n+1} = 4\sin^2\theta_n(1-\sin^2\theta_n)$$

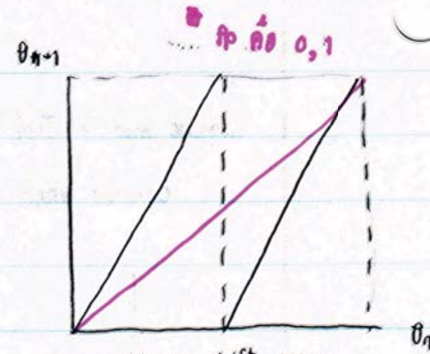
$$= 4\sin^2\theta_n\cos^2\theta_n = (\sin 2\theta_n)^2.$$

$$\Rightarrow \boxed{\theta_{n+1} = \begin{cases} 2\theta_n \\ 2\theta_n - 2 \end{cases}}$$

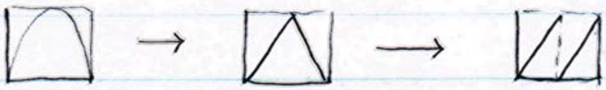
$\hookrightarrow \theta_{n+1} = 2\theta_n \mod 1$



homeomorphism $\longrightarrow$



binary shift map.



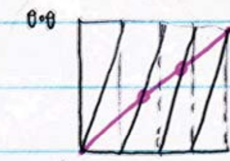
if we understand the last one, then we can understand the first one.

(nonlinear → linear)

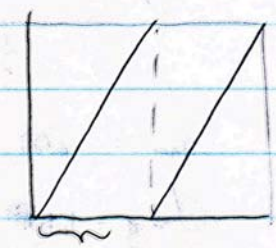
0. 011010... (binary) $\sum_{i=1}^{\infty} a_i \frac{1}{2^i}$ $a_i \in \{0,1\}$

binary expansion.

what about period-2 cycle? → $f(f(\theta))$



periodic orbit found!!



when $\theta \cdot \theta$, this part will expand till 1 and for more than 1 part will be mod.

Dec 3, 2009

» measure the length of east coast
- smaller the ruler, larger the length
- can go to ∞

} do it mathematically

» mathematically

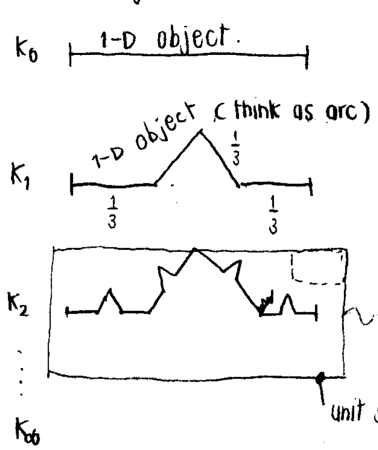
$m=1$, $r=1$ — K_0

$m=4$, $r=3$ — K_1

$m=16$, $r=9$ — K_2

$\vdots$

K_∞



Koch curve $L \sim \left(\frac{4}{3}\right)^n \to \infty$

~ non-differentiable curve (not a function!?)

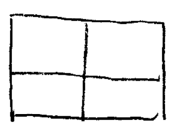
(function is a curve but here is not a function bc it's one to many things. Function is one to one)

~ 1-D is not enough to describe but 2 is too much. it will describle blank space also

» n-dimensional system = need n independent quantities to describe the system sufficiently

» Similarity dimension $= \dfrac{\ln(m)}{\ln(r)}$

$m=1$
$r=1$



$m=4$
$r=2$

$d = \dfrac{\ln 4}{\ln 3} = \dfrac{2\ln 2}{\ln 3} \sim 1.26$ symbols

cantor set

we have no problem for simulate 5, 6, 7. symbols

No. of boxes that there's something from set inside

» Box-counting dimension

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln \frac{1}{\varepsilon}}$$

no. of boxes