Hopf Bifurcation Analysis in Delayed Lienard Equation
-project report for Aero 660

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1 Motivation and Introduction

In this project, we consider a general type of Lienard equation with time-delay and study the hopf bifurcation in such system, the equation is given by

$$\ddot{x}(t) + f(x(t))\dot{x} + g(x(t-\tau)) = 0 \tag{1}$$

where $f,\ g \in C^4,\ f(0) = k > 0,\ g(0) = 0,\ g^{'}(0) = 1,\ \tau > 0$ is a finite time-delay. Note we make the assumption that zero is the only fixed point of this system.

We can rewrite the Equation (1) as

$$\begin{aligned} \dot{x}(t) &= y(t) - R(x(t)) \\ \dot{y}(t) &= -g(x(t-\tau)) \end{aligned} \tag{2}$$

where $R(x) = \int_0^x f(\delta)d\delta$. We can expand Equation (2) in the neighborhood of the null solution in Taylor series

$$\begin{aligned} \dot{x}(t) &= y(t) - kx(t) - \frac{1}{2}f'(0)x^2(t) - \frac{1}{6}f''(0)x^3(t) + O(x^4(t)) \\ \dot{y}(t) &= -x(t-\tau) - \frac{1}{2}g''(0)x^2(t-\tau) - \frac{1}{6}g'''(0)x^3(t-\tau) + O(x^4(t-\tau)) \end{aligned} \tag{3}$$

Note that there are many ways to write Equation (1) in state space form, the reason why we use the form here is that when we are calculating the center manifold, the nonlinearity will only contain the first component of the state vector, which will simplify the calculation.

2 Linear Stability Analysis

The characteristic function of Equation (1) can be obtained by substituting the trial solution $x(t) = e^{\lambda t}$ into its linear part

$$D(\lambda, k) = \lambda^2 + k\lambda + e^{-\lambda\tau} \tag{4}$$

The necessary condition for the existence of a nonzero solution is

$$D(\lambda, k) = 0 \tag{5}$$

On the stability boundary the characteristic equation has one pair of pure imaginary roots. To find these roots, we substitute $\lambda = i\omega,\ \omega > 0$ into Equation (4) and separate the real and imaginary part, we obtain

$$\begin{aligned} \mu^2 - \omega^2 + k\mu + e^{-\mu\tau} \cos \omega\tau = 0 \\ 2\mu + k\omega - e^{-\mu\tau} sin\omega\tau = 0 \end{aligned} \tag{6}$$

Setting $\mu = 0$ can lead us to

$$\begin{aligned} \omega^2 &= \cos \omega\tau \\ k\omega &= \sin \omega\tau \end{aligned} \tag{7}$$

From the above equations we can yield

$$\begin{aligned} k_0 &= \sqrt{\frac{1 - \omega^4}{\omega^2}} \\ \tau_0 &= \frac{1}{\omega} \arctan \frac{k_0}{\omega} \end{aligned} \tag{8}$$

since $k > 0$, we have $0 < \omega < 1$.

Note that actually there are infinite branches of the stability boundaries, but as shown in figure 1, the first branch is the steepest one and it determines the whole stability character of the system.

The obtained $(k_0,\ \tau_0)$ satisfies Equation (7) and gives the boundary of the linearized asymptotic stability for the null solution of Equation (1) in the $(k,\ \tau)$ plane. Then it is left to identify the region in which the roots of Equation (4) have negative real part. Without proof we first introduce a theorem from [2]

Theorem 1 *Assume $a,\ b > 0$. All roots of the equation $(z^2 + az)e^z + b = 0$ have negative real parts if and only if*

$$1 - \frac{bsiny}{ay} > 0 \tag{9}$$

where y is the root of $y = a \cot y$, $0 < y < \pi/2$.

Directly apply theorem 1 to Equation (4) we can get the following results

If $r/k < r_0/k_0$, then all the roots λ of the characteristic Equation (4) have negative real parts. Also the null solution is linearly asymptotically stable.

The necessary condition for the existence of periodic orbits is that by varying the bifurcation parameter k, the critical characteristic roots cross the imaginary axis with positive velocity, i.e. $d\mu/dk\ |_{k=k_0,r=r_0}> 0$.

$$\gamma = \frac{d\mu}{dk}\ |_{k=k_0,r=r_0}= \frac{\omega^2(2+k_0\tau_0)}{(k_0-\tau_0\omega^2)^2+(2\omega+k_0r_0\omega)^2} > 0 \tag{10}$$

The above equation shows that the characteristic roots cross the imaginary axis with positive velocity. In order to show the existence of the periodic orbits we are still left to show that besides a pair of purely imaginary roots, all other roots have negative real part.

From Equation (6) we can obtain the following equation by assuming $\lambda = \mu + i\omega$ is a root of Equation (4)

$$\frac{2\mu + k_0\omega}{\mu^2 - \omega^2 + k_0\mu} = -\tan\omega\tau_0 = -\frac{k_0}{\omega} \tag{11}$$

From this equation we will get

$$\mu(k_0\mu + k_0^2 + 2\omega) = 0 \tag{12}$$

From this equation we can clearly see that the real parts of characteristic roots are either zero or negative.

At this time we can say that there is hopf bifurcation occurring when $r/k = r_0/k_0$.

Then it is natural for us to ask whether the hopf bifurcation is subcritical or supercritical. The following sections will use center manifold theory to ask this question.

3 Operator Differential Equation Formulation

In order to use center manifold to study the infinite dimensional on a two-dimensional center manifold we need the operator differential equation representation of the Equation (1).

This delay differential equation can be expressed as the abstract evolution on the Banach space $\mathcal{H}$ of continuously differentiable functions $\mu:\ [-\tau,\ 0] \to \mathbb{R}^2$

$$\dot{\mathbf{z}}_t = \mathcal{D}\mathbf{z}_t + \mathcal{F}(\mathbf{z}_t) \tag{13}$$

here the $\mathbf{z}_t(\varphi) \in \mathcal{H}$ is defined by the shift of time

$$\mathbf{z}_t(\varphi) = \mathbf{z}(t+\varphi), \quad \varphi \in [-\tau, 0] \tag{14}$$

The linear operator $\mathcal{D}$ at the critical bifurcation parameter assumes the form

$$\mathcal{D}\mathbf{u}(\varphi) = \begin{cases} \frac{d}{d\varphi}\mathbf{u}(\varphi) & \varphi \in [-\tau, 0) \\ \mathbf{L}\mathbf{u}(0) + \mathbf{R}\mathbf{u}(-\tau) & \varphi = 0 \end{cases}$$

while the nonlinear operator is written as

$$\mathcal{F}(\mathbf{u})(\varphi) = \begin{cases} \mathbf{0} & \varphi \in [-\tau, 0) \\ \mathbf{f}(\mathbf{u}) & \varphi = 0 \end{cases}$$

where $\mathbf{f}(\mathbf{u})$ is derived from Equation (3)

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} -\frac{1}{2}f'(0)u_1^2(0) - \frac{1}{6}f''(0)u_1^3(0) \\ -\frac{1}{2}g''(0)u_1^2(-\tau) - \frac{1}{6}g'''(0)u_1^3(-\tau) \end{pmatrix}$$

In order to calculate the center manifold, we need to define an adjoint space $\mathcal{H}^*$ of continuously differentiable functions $\theta : [0, \tau] \to \mathbb{R}^2$ with the adjoint operator

$$\mathcal{D}^*\mathbf{u}(\theta) = \begin{cases} -\frac{d}{d\varphi}\mathbf{u}(\varphi) & \varphi \in (0, \tau] \\ \mathbf{L}\mathbf{u}(0) + \mathbf{R}\mathbf{u}(\tau) & \varphi = 0 \end{cases}$$

we also need to define the bilinear form $(,) : \mathcal{H}^* \times \mathcal{H} \to \mathbb{R}$

$$(\mathbf{v}, \mathbf{u}) = \mathbf{v}^*(0)\mathbf{u}(0) + \int_{-\tau}^{0} \mathbf{v}^*(\delta + \tau)\mathbf{R}\mathbf{u}(\delta)d\delta \tag{15}$$

The former adjoint and the bilinear form provide the basis for a geometry in which it is possible to develop a projection using the basis eigenvectors of the formal adjoint. The advantage of this process is that instead of studying the infinite dimensional delayed system, we can study the two-dimensional center manifold to determine the long time behavior of the full system. For a heuristic argument of how these operators and bilinear form arise, see [1] or more mathematically [2].

Since the critical eigenvalues of the linear operator $\mathcal{D}$ just coincide with the critical characteristic roots of the characteristic function $D(\lambda, k)$, the Hopf bifurcation can be studied at the two-dimensional center manifold embedded in the infinite dimensional phase space.

A first order approximation to this center manifold can be given by the center subspace of the associated linear problem, which is spanned by the real and imaginary parts $\mathbf{p_1}$ and $\mathbf{p_2}$ of the complex eigenfunction $\mathbf{p}(\varphi) \in \mathcal{H}$ corresponding to the critical characteristic root $i\omega$. After separating the real and imaginary parts of the eigenfunction we have the following boundary value problem

$$\begin{aligned} \frac{d}{d\varphi}\mathbf{p}_1(\varphi) &= -\omega\mathbf{p}_2(\varphi) \\ \frac{d}{d\varphi}\mathbf{p}_1(\varphi) &= \omega\mathbf{p}_1(\varphi) \end{aligned} \tag{16}$$

$$\begin{aligned} \mathbf{L}\mathbf{p}_1(0) + \mathbf{R}\mathbf{p}_1(-\tau) &= -\omega\mathbf{p}_2(0) \\ \mathbf{L}\mathbf{p}_2(0) + \mathbf{R}\mathbf{p}_2(-\tau) &= \omega\mathbf{p}_1(0) \end{aligned} \tag{17}$$

The general solution to Equation (16) has the following solution form

$$\begin{aligned} \mathbf{p}_1(\varphi) &= \cos(\omega\varphi)\mathbf{c}_1 - \sin(\omega\varphi)\mathbf{c}_2 \\ \mathbf{p}_2(\varphi) &= \sin(\omega\varphi)\mathbf{c}_1 + \cos(\omega\varphi)\mathbf{c}_2 \end{aligned} \tag{18}$$

The boundary conditions (17) result in a system of linear equations for some of the unknown coefficients

$$\begin{pmatrix} \mathbf{L} + \cos(\omega\tau)\mathbf{R} & \omega\mathbf{I} + \sin(\omega\tau)\mathbf{R} \end{pmatrix} \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \mathbf{0} \tag{19}$$

The center manifold reduction also requires the calculation of the left-hand-side critical real eigenfunctions $\mathbf{q}_{1,2}$ of $\mathcal{D}$ that satisfy the adjoint problem, the procedure are the same with the above calculations, here we directly give the boundary value solution

$$\begin{aligned} \mathbf{q}_1(\theta) &= \cos(\omega\theta)\mathbf{d}_1 - \sin(\omega\theta)\mathbf{d}_2 \\ \mathbf{q}_2(\theta) &= \sin(\omega\theta)\mathbf{d}_1 + \cos(\omega\theta)\mathbf{d}_2 \end{aligned} \tag{20}$$

$$\begin{pmatrix} \mathbf{L}^T + \cos(\omega\tau)\mathbf{R}^T & -\omega\mathbf{I} - \sin(\omega\tau)\mathbf{R}^T \end{pmatrix} \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} = \mathbf{0} \tag{21}$$

Also we need the help of the bilinear form to satisfy 'orthonormality' conditions

$$(\mathbf{q}_1\ ,\ \mathbf{p}_1) = 1, \quad (\mathbf{q}_1\ ,\ \mathbf{p}_2) = 0 \tag{22}$$

Now we need to solve $\mathbf{c}$ and $\mathbf{d}$ to get the $\mathbf{p}$ and $\mathbf{q}$. The above equations can not determine $\mathbf{c}$ and $\mathbf{d}$ uniquely, so we can choose two component of $\mathbf{c}$ freely, i.e. $\mathbf{c}_{11} = 1, \quad \mathbf{c}_{21} = 0$. Then substitute into the boundary conditions we will get

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ k \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 0 \\ \omega \end{pmatrix}$$

$$\mathbf{d}_1 = \Omega \begin{pmatrix} \omega^2(1 + \frac{1}{2}k\tau) \\ -\frac{1}{2}(k - \omega^2\tau) \end{pmatrix} \quad \mathbf{d}_2 = \Omega \begin{pmatrix} \frac{\omega}{2}(k - \omega^2\tau) \\ \omega(1 + \frac{1}{2}k\tau) \end{pmatrix}$$

where we have $\Omega = \frac{1}{(\frac{1}{2}k - \frac{1}{2}\omega^2\tau)^2 + \omega^2(1 + \frac{1}{2}k\tau)^2}$.

Let's decompose the solution $\mathbf{z}_t(\varphi)$ into two components $y_{1,2}$ lying in the center subspace and also into the infinite dimensional component $\mathbf{w}$ transverse to the center subspace

$$\mathbf{z}_t(\varphi) = y_1(t)\mathbf{p}_1(\varphi) + y_2(t)\mathbf{p}_2(\varphi) + \mathbf{w}(t)(\varphi) \tag{23}$$

where we have

$$y_1(t) = (\mathbf{q}_1,\ \mathbf{z}_t)\ |_{\varphi=0}, \quad y_2(t) = (\mathbf{q}_2,\ \mathbf{z}_t)\ |_{\varphi=0}$$

With these new coordinates the operator differential equation (13) can be transformed into a 'canonical form'

$$\dot{y_1} = \omega y_2 + \mathbf{q}_1^T(0)\mathbf{F} \tag{24}$$

$$\dot{y_2} = -\omega y_1 + \mathbf{q}_2^T(0)\mathbf{F} \tag{25}$$

$$\dot{\mathbf{w}} = \mathcal{D}\mathbf{w} + \mathcal{F}(\mathbf{z}_t) - \mathbf{q}_1^T(0)\mathbf{F}\mathbf{p}_1 - \mathbf{q}_2^T(0)\mathbf{F}\mathbf{p}_2 \tag{26}$$

where

$$\mathbf{F} = \mathcal{F}(y_1(t)\mathbf{p}_1(0) + y_2(t)\mathbf{p}_2(0) + \mathbf{w}(t)(0))$$

Note that the nonlinear operator in Equation (26) should be written as

$$\mathcal{F}(y_1\mathbf{p}_1 + y_2\mathbf{p}_2 + \mathbf{w}) = \begin{cases} \mathbf{0} & \varphi \in [-\tau, 0) \\ \mathbf{F} & \varphi = 0 \end{cases}$$

where $\mathbf{F}$ has the form

$$\mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where we can write out f_1 and f_2 as

$$f_1 = -\frac{1}{2}f'(0)(w_1(0) + y_1)^2 - \frac{1}{6}f''(0)(w_1(0) + y_1)^3 \tag{27}$$

$$f_2 = -\frac{1}{2}g''(0)(w_1(-\tau) + y_1)^2 - \frac{1}{6}g'''(0)(w_1(-\tau) + y_1)^3 \tag{28}$$

we neglect the fourth or higher order terms in the above equation.

The purpose of this approach is to study the dynamics of the y_1, y_2 in Equation (24) and (25). As far as now, we know $\mathbf{q}_1$ and $\mathbf{q}_2$. So we still need to know $\mathbf{F} = \mathcal{F}(y_1(t)\mathbf{p}_1(0) + y_2(t)\mathbf{p}_2(0) + \mathbf{w}(t)(0))$, where the only unknown is $\mathbf{w}(t)(0)$. In the next section we will try to approximate $\mathbf{w}$.

4 Two-Dimensional Center Manifold

The center manifold is tangent to the plane in the y_1, y_2 plane at the origin, and it is locally invariant and attractive to the flow of the system (13). Since the nonlinearities considered here are nonsymmetric, we have to compute the second-order Taylor-series expansion of the center manifold. Thus, it can be assumed has the form of truncated power series

$$\mathbf{w}(y_1, y_2)(\varphi) = \frac{1}{2}(\mathbf{h}_1(\varphi)y_1^2 + 2\mathbf{h}_2(\varphi)y_1y_2 + \mathbf{h}_3(\varphi)y_2^2) \tag{29}$$

The time derivative of φ can be expressed by differentiating the right-hand side of the above equation via substituting Equations (24) and (25)

$$\begin{aligned}\dot{\mathbf{w}} &= \mathbf{h}_1 y_1 \dot{y_1} + \mathbf{h}_2 y_2 \dot{y_1} + \mathbf{h}_2 y_1 \dot{y_2} + \mathbf{h}_3 y_2 \dot{y_2} \\ &= \dot{y_1}(\mathbf{h}_1 y_1 + \mathbf{h}_2 y_2) + \dot{y_2}(\mathbf{h}_2 y_1 + \mathbf{h}_3 y_2) \\ &= (\omega y_2 + \mathbf{q}_1^T(0)\mathbf{F})(\mathbf{h}_1 y_1 + \mathbf{h}_2 y_2) + (-\omega y_1 + \mathbf{q}_2^T(0)\mathbf{F})(\mathbf{h}_2 y_1 + \mathbf{h}_3 y_2) \\ &= -\omega \mathbf{h}_2 y_1^2 + \omega(\mathbf{h}_1 - \mathbf{h}_3) y_1 y_2 + \omega \mathbf{h}_2 y_2^2 + O(y^3)\end{aligned} \tag{30}$$

From Equation (26) we can write another form of $\dot{\mathbf{w}}$

$$\dot{\mathbf{w}} = \mathcal{D}\mathbf{w} + \mathcal{F}(\mathbf{z}_t) - \mathbf{q}_1^T(0)\mathbf{F}\mathbf{p}_1 - \mathbf{q}_2^T(0)\mathbf{F}\mathbf{p}_2 \tag{31}$$

Since we have already defined the operators, we can expand the above equation and after comparing the coefficients of the y_1^2, $y_1 y_2$, y_2^2 we can get the governing equations of $\mathbf{h}_1$, $\mathbf{h}_3$ and $\mathbf{h}_3$ together with three boundary conditions, so the next process it to solve the unknown coefficients $\mathbf{h}_1$, $\mathbf{h}_3$ and $\mathbf{h}_3$ in order to write out the $\mathbf{w}(y_1, y_2)(\varphi)$

$$\begin{aligned}\frac{1}{2}\dot{\mathbf{h}}_1 &= -\omega \mathbf{h}_2 + \frac{1}{2}(d_{11}\mathbf{p}_1 + d_{21}\mathbf{p}_2) f_{111} + \frac{1}{2}(d_{12}\mathbf{p}_1 + d_{22}\mathbf{p}_2) f_{211} \\ \dot{\mathbf{h}}_2 &= \omega \mathbf{h}_1 - \omega \mathbf{h}_3 + (d_{11}\mathbf{p}_1 + d_{21}\mathbf{p}_2) f_{112} + (d_{12}\mathbf{p}_1 + d_{22}\mathbf{p}_2) f_{212} \\ \frac{1}{2}\dot{\mathbf{h}}_3 &= \omega \mathbf{h}_2 + \frac{1}{2}(d_{11}\mathbf{p}_1 + d_{21}\mathbf{p}_2) f_{122} + \frac{1}{2}(d_{12}\mathbf{p}_1 + d_{22}\mathbf{p}_2) f_{222}\end{aligned} \tag{32}$$

while we also have the following boundary conditions

$$\begin{aligned}\frac{1}{2}(\mathbf{L}\mathbf{h}_1(0) + \mathbf{R}\mathbf{h}_1(-\tau)) &= -\omega \mathbf{h}_2(0) + \frac{1}{2}\mathbf{m}_0 f_{111} + \frac{1}{2}\mathbf{n}_0 f_{211} - \frac{1}{2}\mathbf{s}_1 \\ \mathbf{L}\mathbf{h}_2(0) + \mathbf{R}\mathbf{h}_2(-\tau) &= \omega \mathbf{h}_1(0) - \mathbf{w}\mathbf{h}_3(0) + \mathbf{m}_0 f_{112} + \mathbf{n}_0 f_{212} - \mathbf{s}_2 \\ \frac{1}{2}(\mathbf{L}\mathbf{h}_3(0) + \mathbf{R}\mathbf{h}_3(-\tau)) &= \omega \mathbf{h}_2(0) + \frac{1}{2}\mathbf{m}_0 f_{122} + \frac{1}{2}\mathbf{n}_0 f_{222} - \frac{1}{2}\mathbf{s}_3\end{aligned} \tag{33}$$

where we have

$$\begin{aligned}\mathbf{m}_0 &= d_{11}\mathbf{p}_1(0) + d_{21}\mathbf{p}_2(0) \\ \mathbf{n}_0 &= d_{12}\mathbf{p}_1(0) + d_{22}\mathbf{p}_2(0)\end{aligned} \tag{34}$$

The coefficients of the quadratic terms in are given by the partial derivatives of f_1 and f_2 evaluated at $y_1 = 0, y_2 = 0$

$$\begin{aligned}f_{111} &= \frac{\partial^2 f_1}{\partial y_1^2} \mid_{\mathbf{0}} = -f'(0) \qquad & f_{211} &= \frac{\partial^2 f_2}{\partial y_1^2} \mid_{\mathbf{0}} = -g''(0) \\ f_{112} &= \frac{\partial^2 f_1}{\partial y_1 y_2} \mid_{\mathbf{0}} = 0 & f_{112} &= \frac{\partial^2 f_2}{\partial y_1 y_2} \mid_{\mathbf{0}} = 0 \\ f_{122} &= \frac{\partial^2 f_1}{\partial y_2^2} \mid_{\mathbf{0}} = 0 & f_{122} &= \frac{\partial^2 f_2}{\partial y_2^2} \mid_{\mathbf{0}} = 0\end{aligned}$$

Introducing the following notation

$$\mathbf{h} = \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix}, \qquad \mathbf{C}_{6\times 6} = \omega \begin{pmatrix} \mathbf{0} & -2\mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & 2\mathbf{I} & \mathbf{0} \end{pmatrix}$$

$$\mathbf{s} = \begin{pmatrix} \mathbf{s}_0\mathbf{s}_1 \\ \mathbf{s}_0\mathbf{s}_2 \\ \mathbf{s}_0\mathbf{s}_3 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} \mathbf{n}_0\mathbf{s}_1 \\ \mathbf{n}_0\mathbf{s}_2 \\ \mathbf{n}_0\mathbf{s}_3 \end{pmatrix}$$

$$\mathbf{s}_1 = \begin{pmatrix} f_{111} \\ f_{211} \end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix} f_{112} \\ f_{212} \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} f_{122} \\ f_{222} \end{pmatrix}$$

$$\mathbf{s}_0 = \begin{pmatrix} d_{11} & d_{12} \\ kd_{11} + \omega d_{21} & kd_{12} + \omega d_{22} \end{pmatrix}, \quad \mathbf{n}_0 = \begin{pmatrix} d_{21} & d_{22} \\ kd_{21} - \omega d_{11} & kd_{22} - \omega d_{12} \end{pmatrix}$$

Now we are ready to write out Equation (32) as the inhomogeneous differential equation

$$\frac{d}{d\varphi}\mathbf{h} = \mathbf{Ch} + \mathbf{s}\cos\omega\varphi + \mathbf{n}\sin\omega\varphi \tag{35}$$

From the classical ordinary equation theory, we know the above equation has the general solution form

$$\mathbf{h}(\varphi) = e^{\mathbf{C}\varphi}\mathbf{K} + \mathbf{M}\cos\omega\varphi + \mathbf{N}\sin\omega\varphi \tag{36}$$

After substitute this solution form into Equation (35) we will get the following equations to solve matrix $\mathbf{M}$ and $\mathbf{N}$

$$\begin{pmatrix} \mathbf{C}_{6\times 6} & -\omega\mathbf{I}_{6\times 6} \\ \omega\mathbf{I}_{6\times 6} & \mathbf{C}_{6\times 6} \end{pmatrix} \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} = -\begin{pmatrix} \mathbf{s} \\ \mathbf{n} \end{pmatrix}. \tag{37}$$

The boundary conditions of (35) can be written in another form as

$$\mathbf{Ph}(0) + \mathbf{Qh}(-\tau) = \mathbf{s} - \mathbf{r} \tag{38}$$

with

$$\mathbf{P} = \begin{pmatrix} \mathbf{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} \end{pmatrix} - \mathbf{C}_{6\times 6}, \quad \mathbf{Q} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{pmatrix}$$

$\mathbf{K}$ is found by substituting the general solution (35) into Equation (38)

$$(\mathbf{P} + \mathbf{Q}e^{-\tau\mathbf{C}})\mathbf{K} = \mathbf{s} - \mathbf{r} - \mathbf{PM} - \mathbf{Q}(\cos\omega\tau\mathbf{M} - \sin\omega\tau\mathbf{N}) \tag{39}$$

From the above equation we can solve $\mathbf{K}$, now we have get the closed form of $\mathbf{h}(\varphi)$ thus the closed form of $\mathbf{w}(\varphi)$, in the center manifold calculation, we are need to know $\mathbf{F}$ where the unknown is $\mathbf{w}_1(0)$ and $\mathbf{w}_1(-\tau)$, notes that we only need to have the first component of $\mathbf{w}$

$$\mathbf{w}_1(0) = \frac{1}{2}((M_1 + K_1)y_1^2 + 2(M_3 + K_3)y_1y_2 + (M_5 + K_5)y_2^2) \tag{40}$$

$$\begin{aligned} \mathbf{w}_1(-\tau) = \frac{1}{2}&((e^{-\mathbf{C}\tau}\mathbf{K}\mid_1 + M_1\cos\omega\tau - N_1\sin\omega\tau)y_1^2 \\ &+ 2(e^{-\mathbf{C}\tau}\mathbf{K}\mid_3 + M_3\cos\omega\tau - N_3\sin\omega\tau)y_1y_2 \\ &+ (e^{-\mathbf{C}\tau}\mathbf{K}\mid_5 + M_5\cos\omega\tau - N_5\sin\omega\tau)y_2^2) \end{aligned} \tag{41}$$

5 Hopf Bifurcation Analysis

In order to restrict a third-order approximation of system (24) and (25) to the two-dimensional center manifold calculated in the previous section, We will assume the dynamics of y_1 and y_2 has the form

$$\begin{aligned} \dot{y}_1 &= \omega y_2 + a_{20}y_1^2 + a_{11}y_1y_2 + a_{02}y_2^2 + a_{30}y_1^3 + a_{21}y_1^2y_2 + a_{12}y_1y_2^2 + a_{03}y_2^3 \\ \dot{y}_2 &= -\omega y_1 + a_{20}y_1^2 + b_{11}y_1y_2 + b_{02}y_2^2 + b_{30}y_1^3 + b_{21}y_1^2y_2 + b_{12}y_1y_2^2 + b_{03}y_2^3 \end{aligned} \tag{42}$$

Using the coefficients above, we can calculate the Poincare-Lyapunov constant $\triangle$

$$\begin{aligned} \triangle = &\frac{1}{8\omega}[(a_{20} + a_{02})(-a_{11} + b_{20} - b_{02}) + (b_{20} + b_{02})(a_{20} - a_{02} + b_{11})] \\ &+ \frac{1}{8}(3a_{30} + a_{12} + b_{21} + 3b_{03}) \end{aligned} \tag{43}$$

The sign of $\triangle$ determines the stability of the Hopf bifurcation. We can also approximate the vibration amplitude by

$$r = \sqrt{-\frac{\gamma}{\triangle}(\tau/k - \tau_0/k_0)}$$

6 Numerical Results

In this section, we studied two type of Lienard equation and check the stability of the hopf bifurcation.

1. Consider the equation

$$\ddot{x}(t) + k(1 - x^2)\dot{x}(t) + x(t-\tau) - Ax^3(t-\tau) = 0 \tag{44}$$

where $k > 0$, A is a nonzero constant, and τ is the time delay. We can easily write out the partial derivative of the $f(x)$ and $g(x)$ evaluated at 0 as:

$f(0) = k,\ f'(0) = 0,\ f''(0) = -2k,\ g(0) = 0,\ g'(0) = 1,\ g''(0) = 0,\ g'''(0) = -6A$ In this example, we fix $k = 1$ to study the effect of the time delay τ to the system. We calculate the boundary from Equation (7) which gives us $\omega = 0.7862$ and $\tau_0 = 1.1506$. Figure $2, 3$ presents the numerical results for the phase portrait when we choose $A = 0.3, \tau = 0.8 < \tau_0$, the Poincare-Lyapunov constant $\triangle = 0.0189 > 0$, so there is subcritical Hopf bifurcation occurs near the origin when τ goes to its critical point. Figure $4, 5$ presents the numerical results for the phase portrait when we choose $A = 0.5, \tau = 0.8 < \tau_0$, the Poincare-Lyapunov constant $\triangle = -0.0250 < 0$, so there is supercritical Hopf bifurcation occurring near the origin when τ goes to its critical point.

2. Consider the following famous sunflower equation which is well know to have supercritical Hopf bifurcation

$$\ddot{x}(t) + k\dot{x}(t) + \sin(x(t-1)) = 0 \tag{45}$$

In this example, we fix $\tau = 1$ to study the effect of the gain k's influence to the system. We calculate the boundary from Equation (7) which gives us $\omega = 0.82413$ and $k_0 = 0.89058$. Figure $6, 7$ presents the numerical results for the phase portrait when we choose $k = 0.6 < k_0$, the Poincare-Lyapunov constant $\triangle = -0.0348 < 0$, so there is supercritical Hopf bifurcation occurs near the origin when k goes to its critical point. Figure 5 presents the numerical results for the phase portrait when we fix $k = 1$, we have $oemga = 0.7862$, $\tau_0 = 0.8476$, then we choose $\tau = 0.5$ the Poincare-Lyapunov constant $\triangle = -0.0377 < 0$, so it's still supercritical Hopf bifurcation occurring near the origin when τ goes to its critical point.

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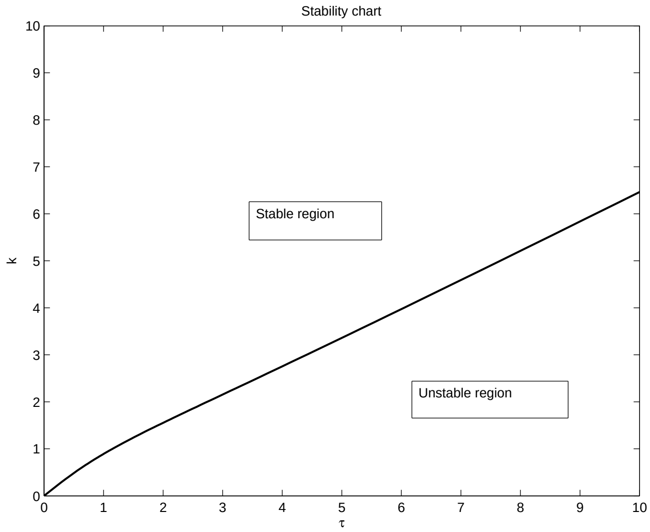

Figure 1: Stability chart for Lienard equation near the origin

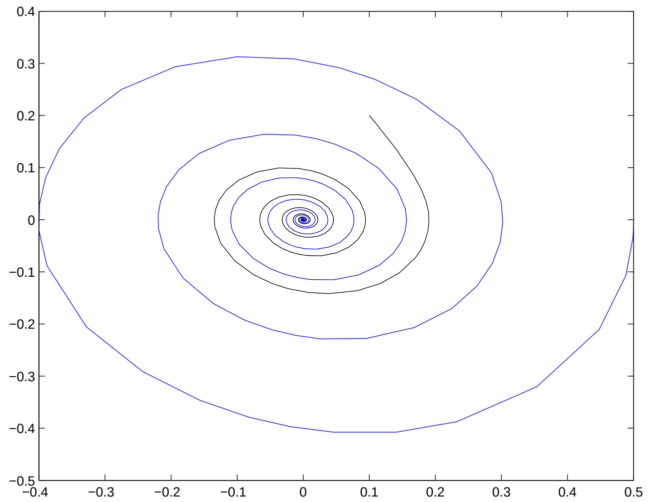

Figure 2: When we setting $A = 0.3, \tau = 0.8 < \tau_0$

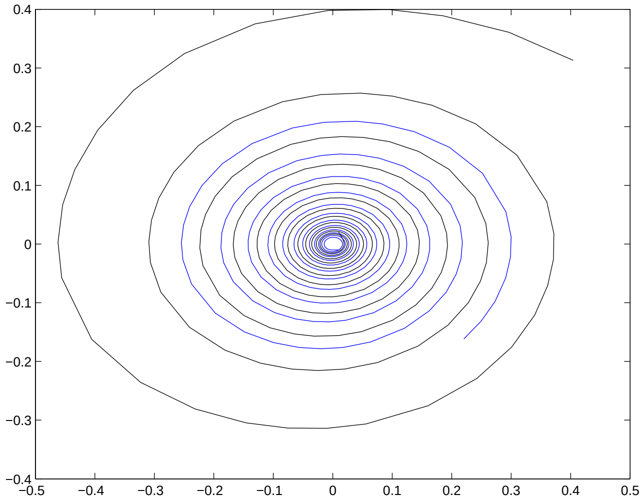

Figure 3: When we setting $A = 0.3, \tau = 1.3 > \tau_0$

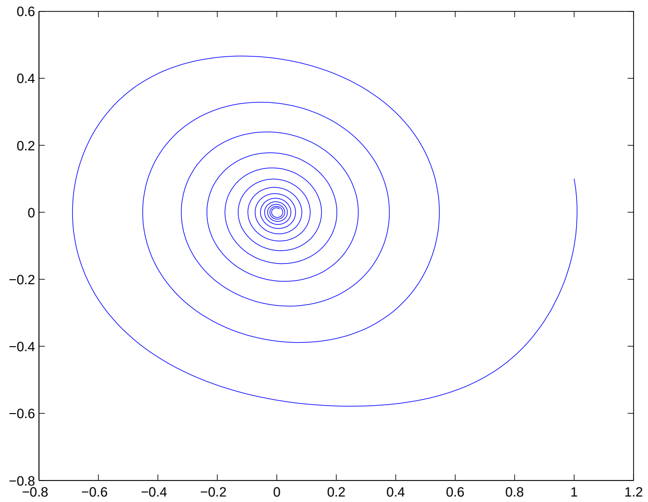

Figure 4: When we setting $A = 0.5, \tau = 1 < \tau_0$

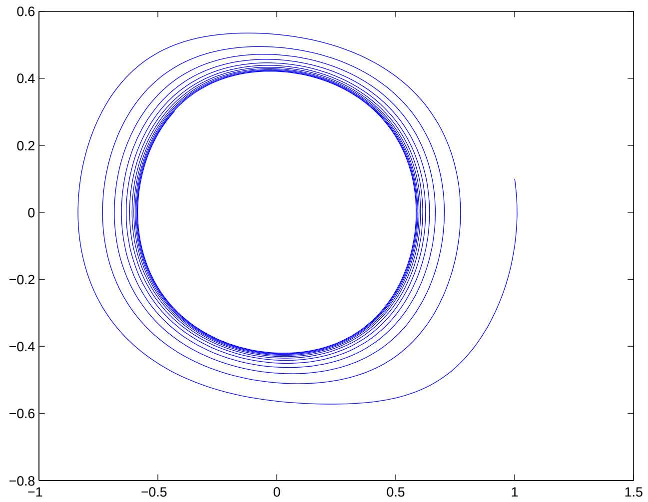

Figure 5: When we setting $A = 0.5, \tau = 1.2 > \tau_0$

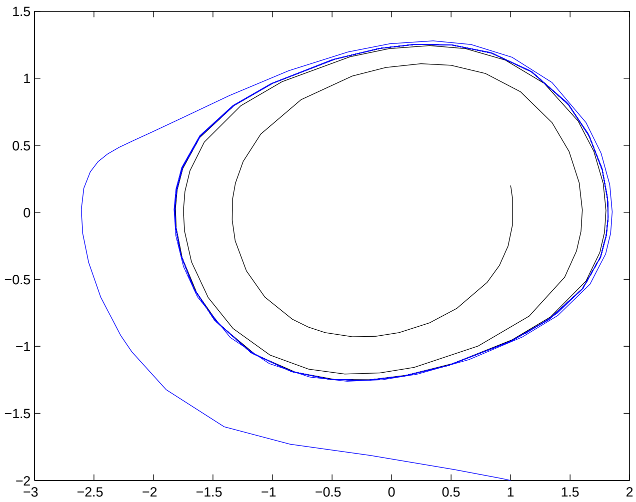

Figure 6: When we setting $k = 0.6 < k_0$

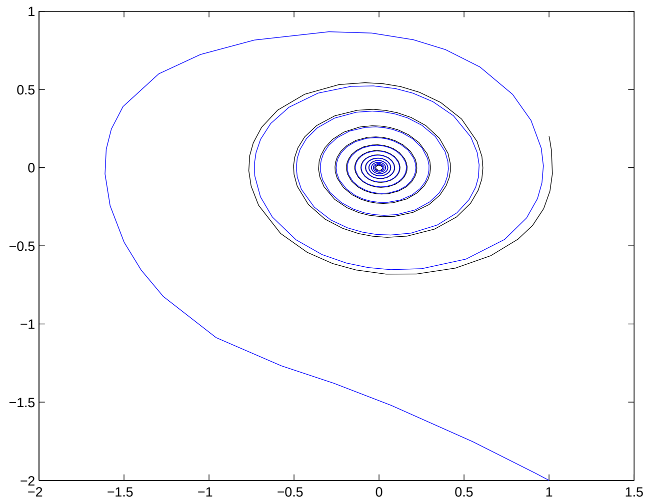

Figure 7: When we setting $k = 1 > k_0$