Dynamical System Analysis of Energy Exchange in Magneto-Hydrodynamic Decaying Isotropic Turbulence

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*Project Report, AERO 660 – Non-Linear Dynamics, fall’08*

1. Introduction

Magneto-hydrodynamics (MHD) studies the dynamics of electrically conducting fluids such as plasmas, liquid metals, liquid superconductors and ionic solutions. In MHD, an external magnetic field induces current in a moving conducting fluid (as per Faradays law, Lenz’s Law), which, in turn, creates body forces in the fluid and also changes the magnetic field itself (a current element creates magnetic field). The governing equations are a coupled system of Navier Stokes equations and Maxwell’s equation of electromagnetism.

At sufficiently high Reynolds number (depends on geometry, dimensions of object), a laminar fluid transitions to a flow (called turbulent flow) which is characterized by random fluctuations (sometimes, of the order of mean flow,), increased mixing and dissipation of momentum and energy. Such a flow is termed ‘turbulent flow’ and has a presence of large range of length and time scales. Energy is transferred from large length scales to small length scales via a process called ‘cascading’. Isotropic turbulence has zero mean velocity () and the product and squares of velocity components and their derivatives are independent of direction or more precisely, invariant of the rotation and reflection of co-ordinate system. Thus, all normal stresses are equal () and tangential (shear) stresses are zero (). In the absence of mean velocity gradients, homogeneous isotropic turbulence decays because there is no production of turbulent kinetic energy. We will refer this as decaying hydrodynamic isotropic turbulence.

In Magnetohydrodynamic homogeneous decaying isotropic turbulence, we have a MHD fluid with an initial isotropic turbulent field in the presence of a magnetic field. The magnetic and kinetic energies interact thorough the Lorentz force. A fluctuating magnetic field increases the vortex stretching and forward cascade (transfer of energy from large scales to small scales) mechanisms. A strong uniform mean magnetic field increases the anisotropy of the turbulent flow field and causes inverse cascading (transfer of energy from small to large scales)

Lorentz force plays an important role in MHD turbulence. Depending upon the magnetic field configuration, the Lorentz forces can alter the coherent structures in turbulent flow by generating flows with or against the hydrodynamic tendency. It can also redistribute energy and momentum in the system between the kinetic and magnetic energy modes.

In the absence of mean velocity gradients, MHD turbulence decays with time (since production, P = 0). Turbulent mixing along with kinetic-magnetic energy interaction (governed largely by Lorentz work) influence the kinetic and magnetic energy decay. Absence of production (source generation), allows both forms of energy to decay independently of hydrodynamic Reynolds number and magnetic Reynolds number  [where,  = magnetic diffusivity].

1. Governing Equations

The governing equations for MHD turbulence are:









where, is the density of the fluid, p is the thermodynamic pressure, B is the intensity of the magnetic field,  is the dynamic viscosity,  is the magnetic permeability of free space and  is the magnetic diffusivity.

The evolution equations for kinetic and magnetic energy are:





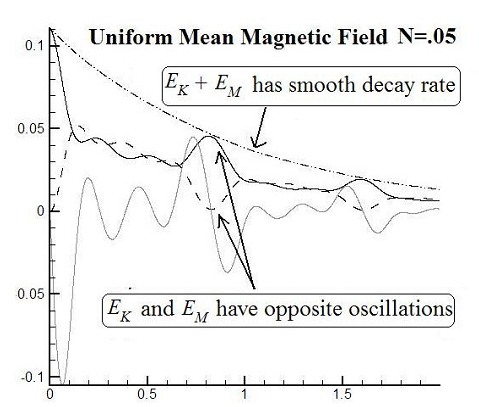


Fig a: Decay of kinetic energy, magnetic energy and total energy for N = 0.03

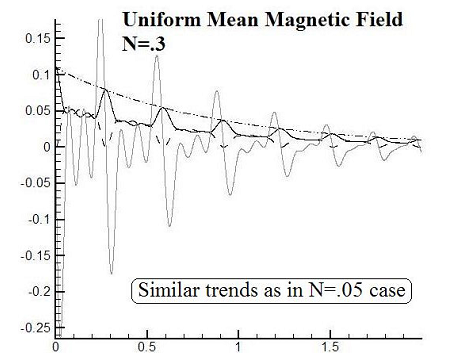


Fig b: Decay of kinetic, magnetic and total energy for N = 0.3

*Fig a, b are adapted from: ‘Richard, J. C., Riley, B. M. and Girimaji S. [1]’*

1. Volume Averaged Kinetic and Magnetic Energy Equation

The instantaneous kinetic (KE) and magnetic energy (ME) equations are volume averaged. Volume averaging removes dependence of KE and ME on spatial coordinates. This gives us an ODE with KE and ME (volume averaged) on the LHS. (Volume averaging → ).

The instantaneous equations for kinetic and magnetic energy are:





,  and, 

Volume averaging for the above set of equations is done as following. Individual terms from the equations would be considered at a time and steps in the derivation using the indicial notation would be demonstrated.

Volume averaging of the Lorenz work term

Consider the indicial notation for the Lorentz work term

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Before we start volume averaging we will consider important results

1. Mass conservation:



1. Solenoidal Magnetic field (divergence free):



1. Homogenous turbulence and magnetic field (spatial derivative of volume averaged terms is zero):



Thus, the volume averaging of the Lorentz work term gives:

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Let us consider the first term in the last equality of the above equation

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Let us consider the second term in the last equality of the above equation

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Thus, the final expression for the volume averaged Lorentz work term is

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Consider the first term on the RHS in the instantaneous magnetic energy equation







Consider the second term on the RHS in the instantaneous magnetic energy equation







Now adding the first two terms on the RHS gives:





Terms A,B and C,D cancel each other. This gives a simplified expression for the two terms as





Volume averaging of the above expression gives





Now let us consider the first term on the RHS in the instantaneous kinetic energy equation.









Volume averaging of the above term gives:





Let us consider the last term on RHS in the instantaneous kinetic energy equation:



Volume averaging gives:



Now let us look at the instantaneous kinetic equation as a whole and perform volume averaging.

Let us define new notations for volume averaged kinetic and magnetic energy:

Volume averaged kinetic energy: 

Volume averaged magnetic energy: 





Similarly, volume averaged magnetic energy equation is:





Finally, the two equations of interest can be written as:

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| --- |
|  |
|  |

The two equations look quite similar. The first term on the RHS is the kinetic dissipation and magnetic dissipation in first and second equation respectively. The second term is same in both equations and is called the Lorentz work term. This is the coupling term which is responsible for transfer of energy from kinetic to magnetic mode.

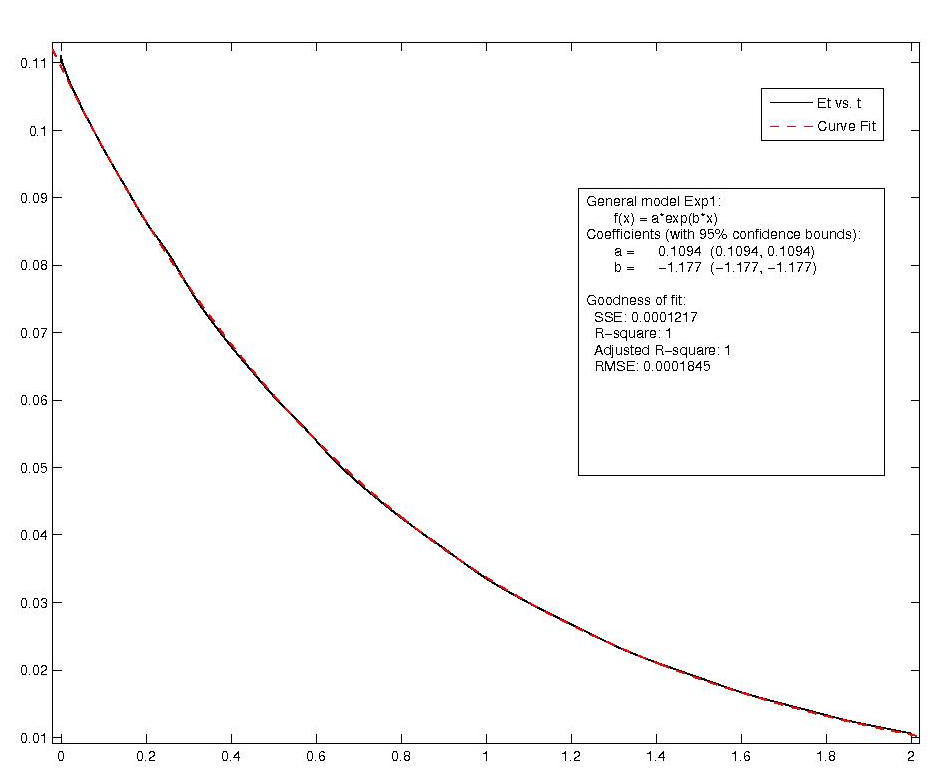
Equation for total energy  is:



The rate of change of total energy is sum of kinetic and magnetic dissipation. The work done by the Lorentz force (exchange term) is canceled.

At a first glance it might seem that if we can represent the dissipation and Lorentz work in terms of volume averaged kinetic and magnetic energy the problem is solved. But, modeling the terms on the right hand side is not easy and needs extensive numerical data and more in-depth understanding of the physics behind decaying isotropic turbulence in a magneto-hydrodynamic fluid.

Let us look at the graph for total energy with time. Total energy decays monotonically with time and looks very close to exponential decay. Hence, curve fitting tool was used to check whether an exponential curve can fit the plot.



Total energy vs. time

Since, total energy decay is exponential, we can postulate the following



The solution for the above ODE is an exponential decay which closely approximates the above graph





Where,

We already have an equation for total energy.



It is really tempting to conclude the following from the above equation after comparing the second and the last equality



Now, let us look at the Lorentz work term

The first attempt to model this term would be dimensional analysis. We start by assuming that the Lorentz work term is an explicit function of kinetic and magnetic energy only and time does not appear explicitly.





Where ‘alpha’ would absorb the rest of the terms.



But, this analysis does not yield other variables that might be present in ‘alpha’ and the value of constants. So, dimensional analysis is of not much significance in the present case.

We would now try different ODE’s using the above knowledge to try to mimic the behavior of the plots obtained from decaying homogenous MHD turbulence.

First, the original plots would be analyzed using ‘Fast Fourier Transform’ and meaningful data would be extracted. First, the kinetic, magnetic and total energy plot with time is shown. It is observed that total energy acts as an envelope for the kinetic and magnetic energy. Using this information the exponential decay is removed from kinetic and magnetic energy plots.

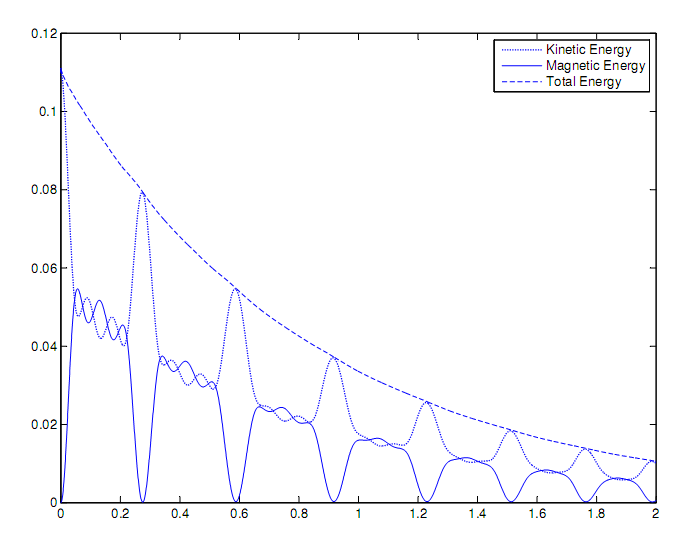


Fig 2 - Kinetic, magnetic and total energy with time in MHD turbulence case with N=0.05

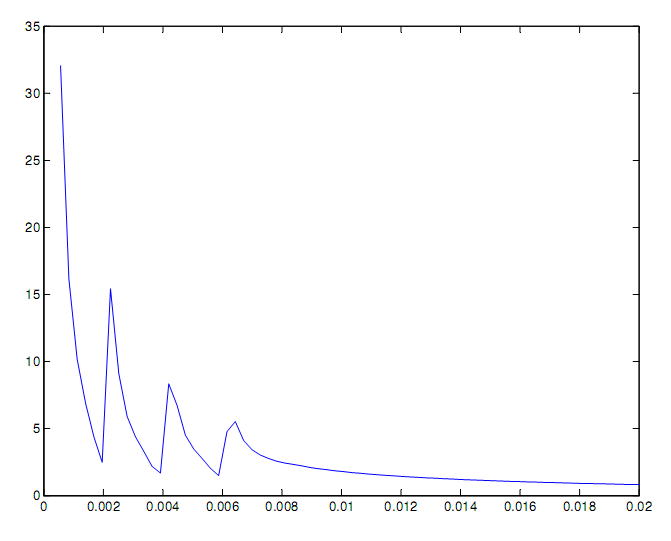
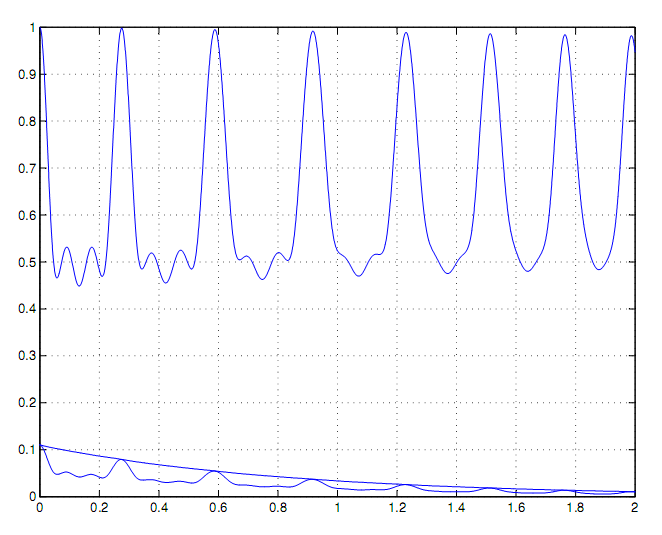
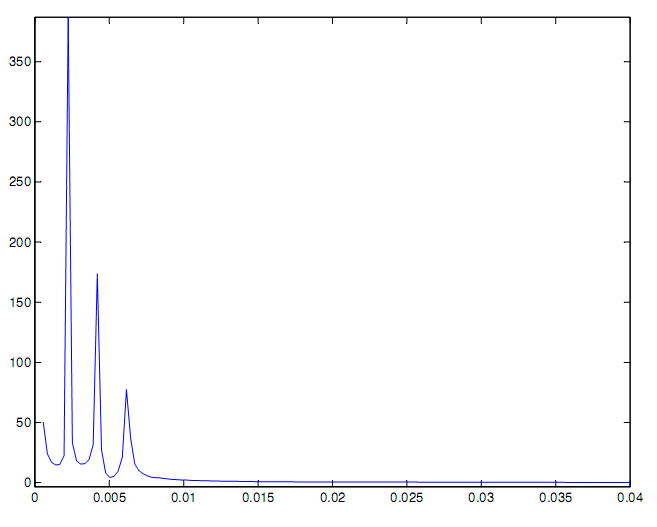


Fig 3: Fast Fourier Transform of kinetic energy (shows three major freq. with substantial spread)



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* Fig 4: Kinetic energy plot (with exponential decay removed) with time.
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* Fig 5: FFT of kinetic energy (with exponential decay removed)

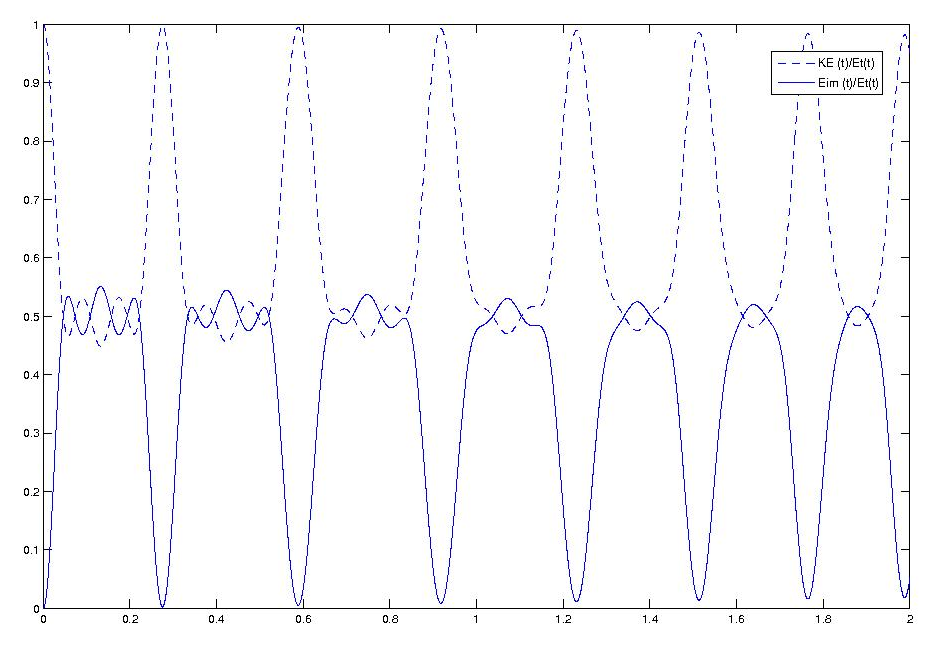


Fig 6: Kinetic and magnetic energy (normalized with total energy) with time

The normalized kinetic and magnetic energy (normalized with total energy at time ‘t’) shows a nice symmetry about the line E = 0.5. We conclude that at times all energy is present in kinetic mode. Then due to the action of Lorentz work, energy is transferred to magnetic mode and very quickly energy is equipartitioned between kinetic and magnetic mode. After few wiggles, Lorentz work again dumps all energy in kinetic mode and the pattern repeats. The difference between successive oscillations is the damping of the intermittent wiggles to an extent that wiggles are totally dissipated.

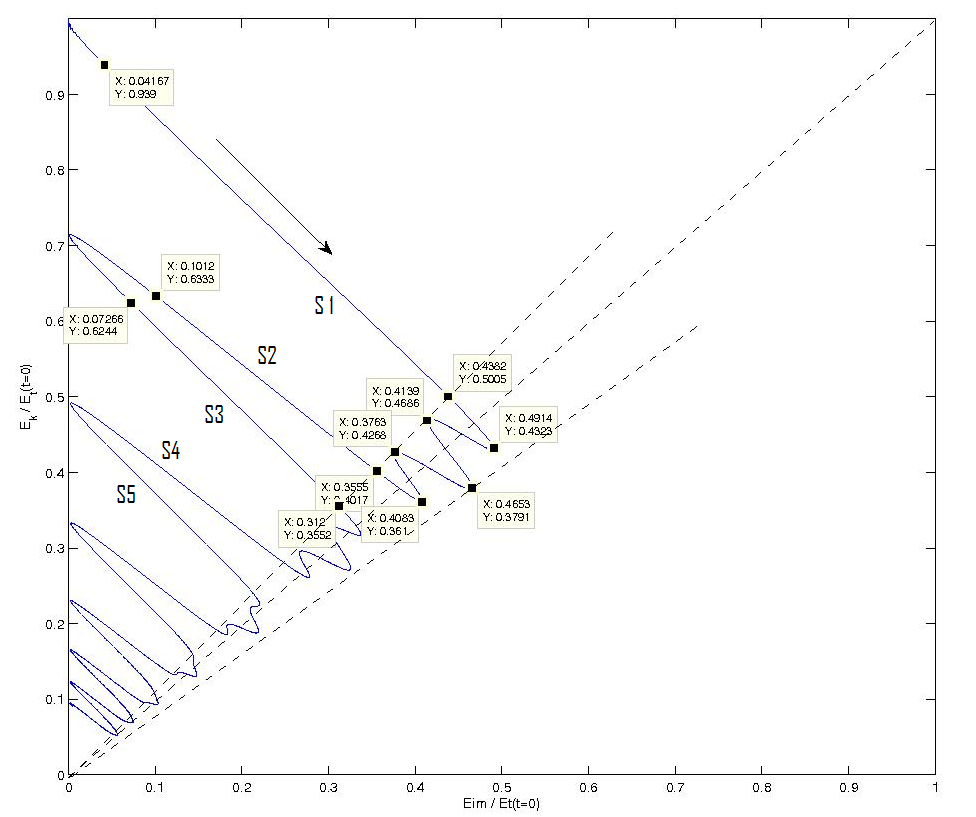


Fig 7: Kinetic energy vs. magnetic energy (both normalized with initial total energy)

Slopes of various lines on the plot are:

S1 = (0.5005 – 0.939) / (0.42382 – 0.04167) = - 1.10584

S2 = (0.4268 – 0.6333) / (0.3763 – 0.1012) = - 0.750636

S3 = - 1.12475

S4 = - 0.88777

S 5 = - 1.13346

This plot is of huge importance as it is the phase portrait for the system of ODE’s. The normalized plot hints towards a similarity solution. Two lines on a phase portrait cannot cross each other due to the arguments of uniqueness. This leads us to conclude that, no matter what initial conditions we start the solution; we would always follow the same curve.

Let us look at non-normalized phase portrait for kinetic and magnetic energy.

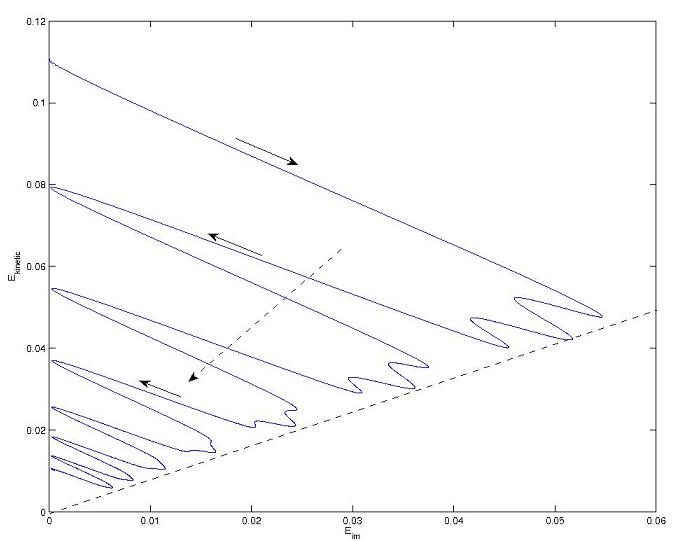


Fig 8: Phase portrait for kinetic and magnetic energy

There is a big conclusion which can be drawn from this plot. If we were to start with a different initial kinetic energy, the curves would cross each other. Also, if we start from another point on the same plot, we won’t follow the same curve. This clearly indicates the presence of an extra dimension. This extra dimension is time ‘t’. This makes us ask a question: “Given a kinetic and magnetic energy, do we have a fixed value of kinetic and magnetic dissipation and Lorentz work?” The answer to the above question is “No, the values would depend on how much time has passed since the flow/simulation was started”. This would mean that we won’t be able to represent kinetic, magnetic dissipation and Lorentz work in terms of kinetic and magnetic energy only. Time would appear explicitly in the system of ODE’s. This can be observed from the following plots too! We have plotted Lorentz work with kinetic and magnetic energy.

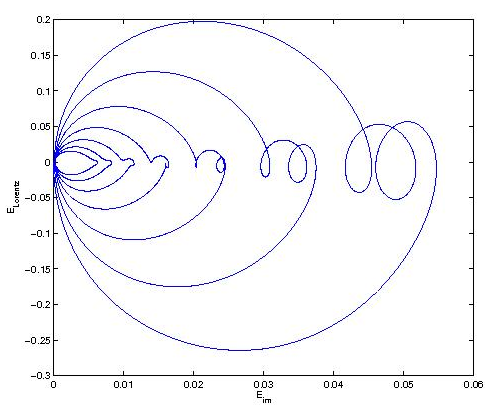


Fig 9: Lorentz work with magnetic energy

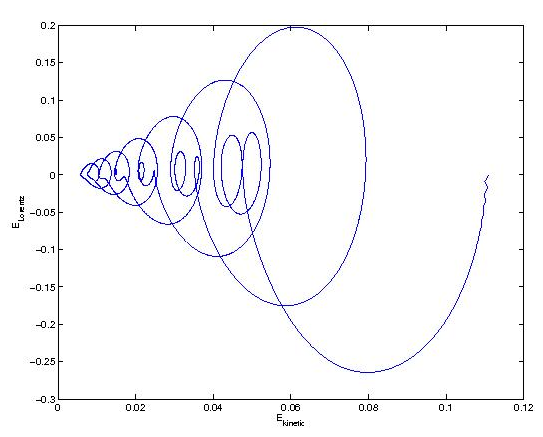


Fig 10: Lorentz work with kinetic energy

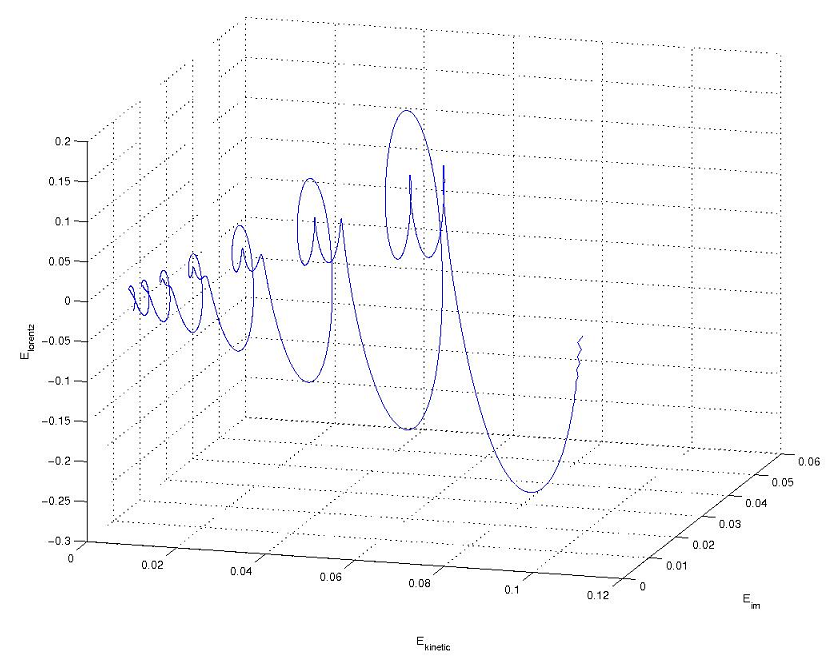


Fig 10: Lorentz work against kinetic and magnetic energy.

Simplified Analysis

In the phase portrait plot for the kinetic and magnetic energy we can clearly identify two regions. One is the straight section where the slope is constant  and the second region where oscillations occur. The oscillatory zone (regions A, B, C, D in fig 11) is confined between two radial lines equispaced from 45° radial line. In this zone the behavior is initially sinusoidal which decays with consecutive occurrences. For the first two oscillatory zones (A, B) we can use the following relation to approximate the behavior:



Although the behavior can be approximated using this for the first few zones (A, B) but for latter zones (C, D) it does not hold well.

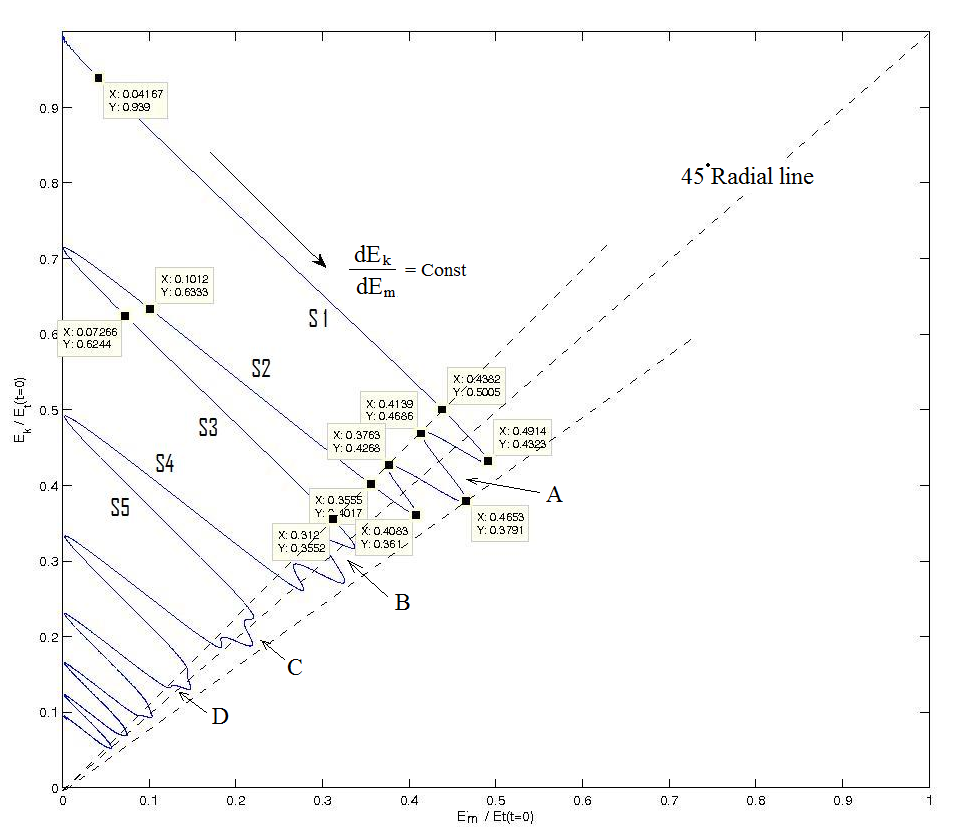


Fig 11: Phase portrait for kinetic and magnetic energy with zones labeled A, B, C & D.

**Rigorous Analysis**

The governing equations for MHD turbulence obtained after volume averaging are:





From the computational data for total energy we know:







where,

From the Fourier transform of the computational data we observe three peaks. Although, there is a spread in the frequencies, we can assume, to a reasonable accuracy, that there are three dominant frequencies present in the problem. This motivates us to choose, after much trial and error, the following function:



This particular function was chosen because the plot looks very similar to the computational data. There is a wide range of parameters at our disposal for fine tuning in order to match the computational results. Apart from this, when *Ek* and *Em* are added, the squares of cosine and sine add up to give a ‘total energy’ that decays exponentially.









where, *ωi* ’s are chosen using FFT and data matching (as FFT only gives the frequency ratios).

Once, we have an expression for *Ek* and *Em,* (as a function of time) we would attempt to obtain an ODE governing it’s evolution in time.

**Attempt – 1**

This is a direct attack over the problem. We take the time derivative of *Ek* and *Em* and try to represent the RHS solely in terms of the primary variables i.e. (*Ek* and *Em*).







Similarly we have,





But, all attempts to write term *A* in terms of *Ek* and *Em* failed. The reason being the presence of *ωi* with the *sine* term.

The second attempt was to break the problem in smaller parts. Let us introduce,



We can easily see that,



Now, we will try to get an ODE for *Eki*’s.















Putting (II) in (I) gives,



Similarly, for magnetic energy:





Now, we have a pair of ODEs which can be solved. Thus, for each i=1, 2 and 3 we will have two ODEs to solve for.





Once we have solved for *Eki*’s and *Emi*’s we can find *Ek* and *Em* using:

Let us try to find the fixed points of the problem







We also know that, 

So, the only fixed points of the problem are .

This makes physical sense too, as the final solution for decaying isotropic turbulence can only be .

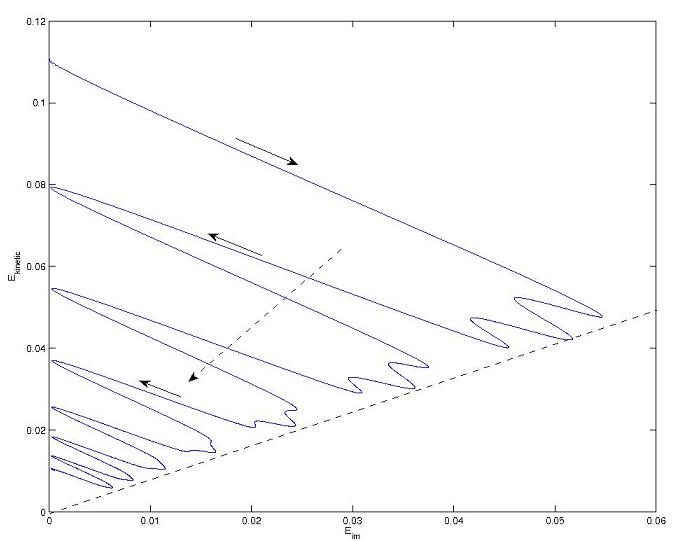


Fig: (0,0) is the fixed point and all solutions finally converge to origin.

**Stability of the origin **





But, the difficulty here is that J(0,0) is undefined as we get a 0/0 case.

The alternative is to construct a Lyapunov function:

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**** is a stable fixed point.

**Improvement on the proposed model**

We observe in FFT plot that amplitudes corresponding to different frequencies are not the same. This would mean the following improvement in the proposed equation for the computational data.





The reasoning behind the addition of the term Δ would be explained later.

We can re-write the above model as:









The derivation of the governing equation from the model is similar to the previous procedure.







We can rewrite the equation for kinetic energy as:



Taking product of equation (III) and (V), we have:



Now, using equation (IV) to replace the exponential term on the LHS we get:







Now, let us take time derivative of the kinetic energy term





Using equation (VI) in the above equation to replace the second term on the RHS we have,



Similarly, for magnetic energy we have the following equation





So, finally the pair of ODEs to be solved for each i = 1, 2 and 3 are:



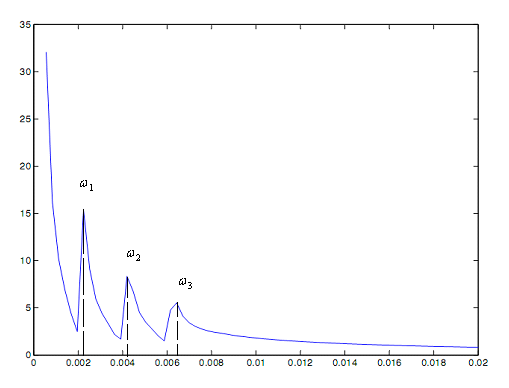


The initial conditions are:



The above model also verifies our previous postulate



Fast Fourier Transform of kinetic energy (shows three major freq. with substantial spread)

Let us try to extract data from the FFT plot

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From total energy decay plot we can find the decay constant:



When we try to match the model with the computational data we get the exact values for the frequency.

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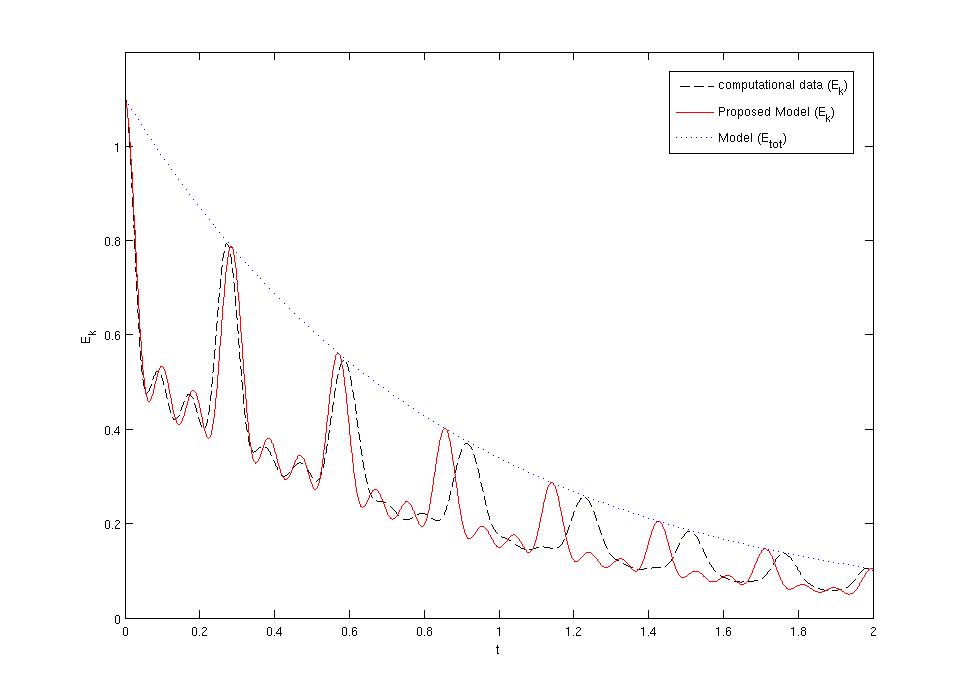


Fig: comparison of computational data with proposed model

The model matches the data for first two peaks very nicely and then there is a lag. This lag could be attributed to the spread in the frequencies in the FFT, which we neglected and choose the three prominent peaks.

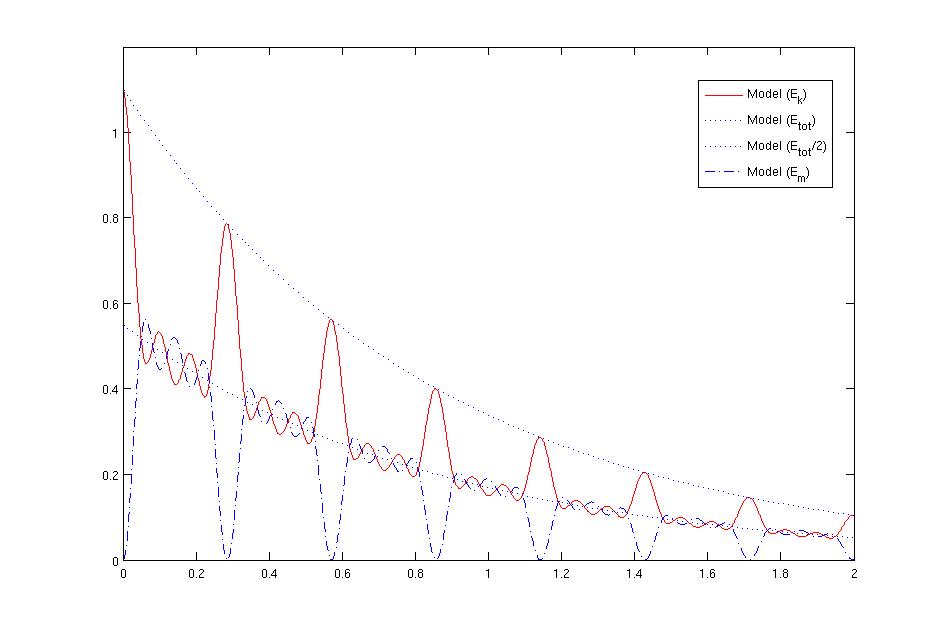


Fig: Kinetic and Magnetic Energy plot from the model.

Now, based on our proposed model we would like to prove that at any given instant kinetic and magnetic energy are equally spaced from half of the total energy i.e.:



Where, θ can have + ve or – ve value. We start with the expression for total kinetic and magnetic energy:









Now,







For magnetic energy, we have:







Let us define θ as following:



This allows us to write equations (A1) and (A2) in the following form:



This leads us to the final discussion. Although there is a continuous exchange of energy between kinetic and magnetic mode but the energy is not equally partitioned. There is a part of kinetic energy that is not exchanged. It is very easy to see that, it is the term involving Δ.



The percentage of un-exchanged energy is:



From actual data, the computed value is:



References

1. Richard, J. C., Riley, B. M. and Girimaji, S.: ‘Energy Exchange and Vortex-Stretching in Magnetohydrodynamic Decaying Isotropic Turbulence’, Texas A&M University – Paper in writing.

2. Pope, S. B.: ‘Book: Turbulent Flows’, Cambridge University Press, 2000.

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