Proposed Title:
Investigation of Feedback Delay in the van der Pol Oscillator

Lesley A. Weitz

I. Introduction

Limit cycles are periodic oscillations that arise in nonlinear systems even in the absence of a periodic forcing input. The standard limit cycle of a nonlinear system can be described by a period, waveform, and amplitude, and any perturbation from a stable limit cycle will return to the standard limit cycle.[1] Limit cycles can occur in different types of dynamical systems including biological systems, airplane-wing flutter, and bridge design. In many of these cases, there is a need to control or modify limit-cycle behavior in order to avoid dangerous periodic behavior. Research seeks to answer the question: how can different feedback-control designs modify the limit cycle period, waveform, and amplitude?

In actual implementation of feedback control, stability challenges may occur due to inherent delays in state feedback due to measurement and sensing delays. These delays may need to be taken into account when evaluating closed-loop system stability. This project will evaluate how delayed feedback affects limit-cycle behavior.

II. Background

The van der Pol Oscillator, shown below, is a classical example of a nonlinear dynamical system that displays limit cycle oscillations. Therefore, this model serves as a prototypical example in which to investigate control design to avoid dangerous limit-cycle behavior.

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = f(t), \qquad \epsilon > 0$$

When the forcing input, $f(t)$, is zero, there is a stable limit cycle that attracts all solutions starting at $(x(0), \dot{x}(0))$ except at the origin.[1,2] For the *strongly nonlinear damping* case or $\epsilon >> 1$, the limit cycle displays two time scales. One time scale is related to a slow buildup and the second scale is related to a sudden discharge. In the *weakly nonlinear damping* case or $\epsilon << 1$, the limit cycle behavior closely approximates a simple harmonic oscillator: $\ddot{x} + x = 0$.

Here, we will investigate feedback control of the form $f(t) = k_1 x(t-\tau) + k_2 \dot{x}(t-\tau)$, which leads to a delay-differential equation (DDE).[3,4] Atay has investigated feedback of the form $f(t) = \epsilon k x(t-\tau)$, where position feedback with delay can affect both the amplitude and frequency of the limit cycle.[2] He analyzed the nonlinear DDE using the method of averaging to find decoupled differential equations in polar coordinates from which an attracting periodic solution was determined.[5,6]

III. Approach and Proposed Solution

It is proposed that this project have two parts. Firstly, I would like to reproduce the results from the Atay research in order to fully understand the application of the method of averaging to the nonlinear DDE. For this part of the project, I plan to analyze the behavior of the limit cycle for the different control parameters (gains and time-delay), and more specifically, I plan to characterize the bifurcations that occur for the parameter changes. Atay has noted that this particular approach is valid only for the weakly nonlinear case.

Secondly, I would like to understand and investigate the application of perturbation techniques to the van der Pol oscillator with delayed feedback. This path is less well-defined as more research is needed to determine the possible solution approaches. Some initial investigation has indicated that regular perturbation theory leads to errant approximations of the limit cycle.[1] Strogatz suggests another approach called *two-timing*[a], which uses the knowledge that the van der Pol oscillator has two time scales that are related to the slow build-up and fast discharge behavior of the oscillator. I plan to investigate this approach for the DDE. In particular, I would like to determine whether this approach is applicable to both the strongly and weakly nonlinear cases.

Originally, I had an interest in using perturbation theory for equations with small time lags as described by Bellman.[7] Bellman assumes a linear nth-order DDE form with a small time lag. He then expands the DDE about the time lag accounting for only first-order effects (neglecting terms of $0(\epsilon^2)$ to derive an $(n{+}1)$th ODE from which stability can be analyzed. I hope to investigate this method of analyzing feedback with delay; however, two challenges come to mind. One challenge is that this approach is developed for linear DDEs, and the van der Pol oscillator is definitely nonlinear. A second challenge is due to the failures of the regular perturbation methods mentioned above; therefore, this approach may need to be investigated using a two-timing approach.

References

[1]Strogatz, S. H., *Nonlinear Dynamics and Chaos*, Perseus Books Publishing, Cambridge, MA, 1994, Chapter 7.

[2]Atay, F. M., "Van der Pol's Oscillator under Delayed Feedback," *Journal of Sound and Vibration*, Vol. 218, 1998, pp. 333–339.

[3]Driver, R. D., *Ordinary and Delay Differential Equations*, Springer-Verlag, New York, NY, 1977, Chapter 5.

[4]Hale, J., *Theory of Function Differential Equations*, Springer-Verlag, New York, NY, 1977.

[5]Hale, J. K., "Averaging Methods for Differential Equations with Retarded Arguments annd a Small Parameter," *Journal of Differential Equations*, Vol. 2, 1966, pp. 57–73.

[6]Lehman, B. and Weibel, S. P., "Averaging Theory for Functional Differential Equations," *Proceedings of the 37th IEEE Conference on Decision & Control*, Tampa, FL, 1998, pp. 1352–1357.

[7]Bellman, R., *Perturbation Techniques in Mathematics, Engineering, and Physics*, Holt, Rinehart, and Winston, Inc., New York, NY, 1966.

[a]It turns out that two-timing is not just a mean thing to do to your significant other